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ERRATA.

In Diagram No. 2 of Mr. Spens' paper—For 1861, per Table II., read 1861, per Table XI.

Page 74, last line of figures—For ·928 and ·912, read 2·160 and 2·200.

Page 76, heading of last column, Table VIII.—For Table II., read Table VI.

Page 297, line 18—For ·002758, read ·01379.

Page 298, line 5—For ·006784, read ·033919.

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Seventh Census of England and Wales, April, 1861. *Summary of General Results.**

THE Act for taking the Census of England required that the 31,000 Enumerators employed should copy into as many books the householders' schedules and other particulars collected by them in their several districts. These books were to be placed, with the schedules, in the hands of the 2,197 Registrars, who were to subject them to a strict examination, and make all necessary corrections. This being accomplished, the books and other documents were to be transferred before the 30th April to the custody of the 631 Superintendent Registrars, who were required to test the accuracy of their contents by a further process of revision.

The Superintendent Registrars were allowed a short time for the purpose of returning the revised books to the Census Office, where they have still to undergo strict and minute revision, before any *detailed* and *exact* statement of results can be given. As this

* This abstract contains the substance of one laid before Parliament on the 7th June last, and prepared in the Registrar-General's office by Dr. Farr and Mr. Hammack. It may be thought by some inexpedient to occupy the pages of this *Journal* with matter so easily accessible elsewhere; but we believe most of our readers will consider it only fitting that some brief record should appear in it of the results of an inquiry of such importance as this undoubtedly is.—ED. *A. M.*

essential labour must, however, of necessity, engage much time, it has been thought desirable not to withhold from the public such an approximation to the general results as might be obtained, without waiting for the entire completion of this series of checks. The Registrars, therefore, were desired to forward to the Census Office summaries of the population and houses within their respective districts.

From these summaries the following tables have been chiefly compiled, and they must be taken to represent the results of the Census according to the statements of the *local officers,* previous to the revision now in progress at the central office. And although minute accuracy is not to be looked for, yet it is apprehended that the alterations which a careful revision of the original documents may render necessary will not be of importance sufficient to lessen the value of the figures as materials for whatever *general* inferences may fairly be drawn from them.

Table I. comprises the *Population enumerated* on April 8th, 1861, in England and Wales and the islands in the British seas, amounting to 20,205,504. The portion of the Army, Royal Navy, and Merchant Seamen *out of the country* at the time of the Census is not included; and as it appears from official returns that the *Army abroad* amounted to 137,000, the *Royal Navy and Marines absent* from the United Kingdom to about 42,900, and the *Merchant Seamen absent* on voyages to about 96,000, it may be assumed that the numbers of these classes *belonging to* England and Wales were collectively not less than 162,021, mostly adult males.

Table II. exhibits the number of *Houses* and the *Population enumerated* in England and Wales in 1851 and 1861, with the *increase* in the interval.

Table III. shows the population of England and Wales at each of the Censuses, 1801-61, *including the Army, Navy, and Merchant Seamen abroad* as well as at home, with the increase in each decennial period. The *actual increase* of population (2,169,576), between 1851 and 1861, was greater than in any previous decennial period, but the *rate of increase,* owing to active emigration, had somewhat diminished.

Table IV. presents a comparative view of the number of houses enumerated at each of the Censuses.

Table V. shows the *Houses* and *Population* enumerated in the Counties (proper) in 1851 and 1861, together with the increase or decrease of persons in each county in the period between the enumerations.

In Table VI. the numbers of *Inhabited Houses* and of the *Population* in 1851 and 1861, with the increase or decrease of persons in the interval, are presented for each of the 631 Superintendent Registrars' Districts into which England is divided. These districts are for the most part co-extensive with single Poor Law Unions, or two or more unions combined, but in some instances they consist of places in which the relief of the Poor is regulated under local Acts or otherwise than by the Poor Law Amendment Act. A column is added in this table, showing the excess of *Registered Births* over *Deaths* in the ten years 1851-60; but, in comparing these numbers with the *ascertained* increase or decrease, it is necessary to bear in mind that the whole of the births are not registered. No birth can be recorded after six months, and registration is not compulsory; so, in the present state of the law, many births are not entered in the registers, especially in London and the large towns.

Table VII. is framed from the details in Table VI. The Superintendent Registrars' Districts are grouped together in eleven divisions, each comprising the whole, or nearly the whole, of the several counties named. In the columns showing the ascertained increase of population, and the excess of births over deaths in these divisions, may be traced the powerful stream of immigration into the principal centres of trade and the seats of mining and manufacturing industry. Thus in London (Division I.), where the excess of registered births over deaths was 253,989, the influx of persons from other parts had raised the actual increase to 440,798; in the Eastern Division (IV.), consisting of the counties of Essex, Suffolk, and Norfolk, the ascertained increase was only 28,220, while the natural increase or excess of births over deaths was 129,726.

In Table VIII. the *Houses* and *Population* in the principal *Cities* and *Boroughs* in 1861 are stated, chiefly on the authority of the Superintendent Registrars in whose districts the places are wholly or partly comprised. The municipal and parliamentary boundaries are frequently not conterminate with those of parishes

or other local subdivisions, and great care is required in assigning the population within their exact limits. It is probable that sufficient attention to questions of boundary has not been given in some cases by the local officers in preparing these statements within the limited time allowed them. The numbers, must, therefore, be received as approximations only, subject to revision hereafter at the Census Office. This table includes only corporate towns and boroughs returning members to Parliament; several large towns, consequently, do not appear in it.

Tables IX. and X., referring to the Channel Islands and the Isle of Man, require no explanation.

Table XI. relates to *Emigration*. According to the returns of the Emigration Commissioners, 2,249,355 emigrants sailed from the ports of the United Kingdom in the interval between the Census of March 31st, 1851, and the Census of April 8th, 1861. But 194,532 of the number were probably of foreign origin, leaving 2,054,823 emigrants from the population of the United Kingdom; of whom about 640,210 were of English origin, 183,627 were of Scotch origin, and 1,230,986 were of Irish origin.

TABLE I.—*England and Wales and Islands in the British Seas. Population Enumerated in England and Wales, and in the Islands in the British Seas, on April 8th,* 1861.

	Persons.	Males.	Females.
Population enumerated on April 8th, 1861*	20,205,504	9,825,246	10,380,258
England and Wales	20,061,725	9,758,852	10,302,873
Islands in the British Seas	143,779	66,394	77,385

* "This number includes the part of the Army in England and the Channel Islands; also the Navy, Merchant Seamen, and others on board vessels in the ports, rivers, and creeks, on the 8th April. The part of the Army, Navy, and Merchant Seamen out of the country, is *not included*."

TABLE II.—*England and Wales. Houses and Population enumerated in England and Wales in* 1861 *and* 1851.

1	2	3	4	5	6	7
Census Year.	Houses.			Population.		
	Inhabited.	Uninhabited.	Building.	Persons.	Males.	Females.
1861 (April 8th)	3,745,463	182,325	27,580	20,061,725	9,758,852	10,302,873
1851 (March 31st)	3,278,039	153,494	26,571	17,927,609	8,781,225	9,146,384
Increase in the interval between 1851 and 1861	467,424	28,831	1,009	2,134,116	977,627	1,156,489

TABLE III.—*England and Wales. Population at each of the Censuses* 1801-61, *including the Army, Navy, and Merchant Seamen Abroad, as well as at Home, belonging to England and Wales, Actual Increase, and Rates of Increase, in the Decennial Periods.*

	1801.	1811.	1821.	1831.	1841.	1851.	1861.
Population enumerated	9,156,171	10,454,529	12,172,664	14,051,986	16,035,198	18,054,170	20,223,746

	1801–11	1811–21	1821–31	1831–41	1841–51	1851–61
Actual Increase in the Decennial Periods	1,298,358	1,718,135	1,879,322	1,983,212	2,018,972	2,169,576
Decennial Rates of Increase per cent.	14	16	15	14	13	12

TABLE IV.—*England and Wales. Number of Houses Enumerated at each of the Censuses* 1801-61.

	1801.	1811.	1821.	1831.	1841.	1851.	1861.
Houses—							
Inhabited	1,575,923	1,797,504	2,088,156	2,481,544	2,943,945	3,278,039	3,745,463
Uninhabited	57,476	51,020	69,707	119,915	173,247	153,494	182,325
Building	No return	16,207	19,274	24,759	27,444	26,571	27,580

Table V.—*England and Wales. Houses and Population*

1	2	3	4	5	6	7	8
		1851 (March 31st).					
		Houses.			Population.		
Counties (Proper).	Area in Statute Acres.	Inhabited.	Uninhabited.	Building.	Persons.	Males.	Females.
England & Wales	37,324,915	3,278,039	153,494	26,571	17,927,609	8,781,225	9,146,384
ENGLAND.							
Bedford	295,582	24,673	661	127	124,478	59,941	64,537
Berks	451,040	33,481	1,397	197	170,065	84,927	85,138
Buckingham	466,932	33,196	1,206	98	163,723	81,074	82,649
Cambridge	523,861	37,226	1,629	195	185,405	92,699	92,706
Chester	707,078	85,260	4,341	845	455,725	222,386	233,339
Cornwall	873,600	67,987	4,544	347	355,558	171,636	183,922
Cumberland	1,001,273	36,763	1,545	239	195,492	96,244	99,248
Derby	658,803	59,371	2,498	453	296,084	147,737	148,347
Devon	1,657,180	98,387	6,014	751	567,098	269,583	297,515
Dorset	632,025	36,138	1,587	215	184,207	89,204	95,003
Durham	622,476	64,977	2,794	570	390,997	196,700	194,297
Essex	1,060,549	73,530	3,569	381	369,318	185,399	183,919
Gloucester	805,102	86,359	5,318	441	458,805	218,187	240,618
Hereford	534,823	23,890	1,191	77	115,489	58,114	57,375
Hertford	391,141	32,573	1,188	207	167,298	82,831	84,467
Huntingdon	230,865	13,285	632	64	64,183	31,933	32,250
Kent	1,041,479	107,748	5,460	1,267	615,766	307,041	308,725
Lancaster	1,219,221	349,938	17,420	3,463	2,031,236	991,090	1,040,146
Leicester	514,164	48,953	1,629	211	230,308	112,937	117,371
Lincoln	1,776,738	81,335	3,450	592	407,222	205,083	202,139
Middlesex	180,168	239,362	11,874	3,392	1,886,576	882,823	1,003,753
Monmouth	368,399	28,939	1,353	152	157,418	82,349	75,069
Norfolk	1,354,301	93,143	3,505	452	442,714	215,254	227,460
Northampton	630,358	43,942	1,538	227	212,380	105,984	106,396
Northumberland	1,249,299	47,737	2,064	386	303,568	149,515	154,053
Nottingham	526,076	55,019	1,502	250	270,427	132,263	138,164
Oxford	472,887	34,398	1,334	105	170,439	85,524	84,915
Rutland	95,805	4,588	153	14	27,983	11,801	11,182
Salop	826,055	45,648	2,062	116	229,341	114,340	115,001
Somerset	1,047,220	85,054	4,912	393	443,916	211,045	232,871
Southampton	1,070,216	75,238	3,543	613	405,370	202,014	203,356
Stafford	728,468	116,273	4,668	958	608,716	310,032	298,684
Suffolk	947,681	69,282	3,107	449	337,215	166,308	170,907
Surrey	478,792	108,822	5,770	1,540	683,082	325,041	358,041
Sussex	934,851	58,663	2,247	606	336,844	165,772	171,072

enumerated in the Counties (proper) in 1851 *and* 1861.

9	10	11	12	13	14	15	16	17
1861 (April 8th).						Increase or Decrease in the Number of Persons between 1851 and 1861.		Counties (Proper).
Houses.			Population.					
Inhabited.	Uninhabited.	Building.	Persons.	Males.	Females.	Increase.	Decrease.	
3,745,463	182,325	27,580	20,061,725	9,758,852	10,302,873	2,134,116	—	England and Wales
								ENGLAND.
27,419	705	142	135,265	63,780	71,485	10,787	—	Bedford
35,880	1,335	202	176,103	86,657	89,446	6,050	—	Berks
34,680	1,287	285	166,597	82,148	84,449	2,878	—	Buckingham
37,677	1,847	76	175,950	86,304	89,646	—	9,455	Cambridge
97,952	5,305	751	505,153	243,960	261,193	49,428	—	Chester
73,243	3,389	469	369,323	176,107	193,216	13,765	—	Cornwall
40,579	2,061	251	205,293	100,218	105,075	9,801	—	Cumberland
69,404	3,318	531	339,377	170,509	168,868	43,293	—	Derby
101,406	5,459	678	584,531	279,216	305,315	17,433	—	Devon
37,745	1,531	280	188,651	91,544	97,107	4,444	—	Dorset
84,877	4,164	588	509,018	258,343	250,675	118,021	—	Durham
81,220	4,092	538	404,644	202,713	201,931	35,346	—	Essex
93,900	4,711	565	485,502	228,312	257,190	26,697	—	Gloucester
25,271	800	171	123,659	62,753	60,906	8,170	—	Hereford
34,899	1,535	188	173,294	84,242	89,052	5,996	—	Hertford
13,733	497	53	64,297	31,713	32,584	114	—	Huntingdon
126,246	5,138	1,055	733,675	368,450	365,225	117,909	—	Kent
439,634	19,831	3,703	2,428,744	1,171,322	1,257,422	397,508	—	Lancaster
51,909	2,454	230	237,402	115,179	122,223	7,094	—	Leicester
86,688	4,104	523	411,997	204,200	207,797	4,775	—	Lincoln
279,831	13,407	3,240	2,205,711	1,020,191	1,185,580	310,195	—	Middlesex
33,101	2,031	264	174,670	89,533	85,137	17,252	—	Monmouth
96,951	4,929	354	435,422	208,910	226,512	—	7,292	Norfolk
48,547	2,114	283	227,727	112,963	114,764	15,347	—	Northampton
55,900	2,635	683	343,028	170,359	172,669	39,460	—	Northumberland
62,557	4,432	500	293,784	141,027	152,757	23,357	—	Nottingham
36,309	1,314	220	172,266	85,386	86,880	1,827	—	Oxford
4,652	175	17	21,851	10,899	10,960	—	1,124	Rutland
48,155	1,572	221	240,876	120,154	120,722	11,535	—	Salop
87,561	4,009	559	444,725	209,301	235,424	809	—	Somerset
86,494	3,707	637	481,495	246,076	235,419	76,125	—	Southampton
147,244	8,871	1,109	746,584	376,464	370,120	137,868	—	Stafford
73,067	3,662	219	336,271	164,239	172,032	—	964	Suffolk
130,563	4,611	1,384	830,685	392,786	437,899	147,603	—	Surrey
65,471	2,754	430	363,648	174,782	188,866	26,804	—	Sussex

Table V.—*England and Wales.*

1	2	3	4	5	6	7	8
		1851 (March 31st).					
		Houses.			Population.		
Counties (Proper).	Area in Statute Acres.	Inhabited.	Uninhabited.	Building.	Persons.	Males.	Females.
Warwick	563,946	96,731	4,596	992	475,013	232,411	242,602
Westmoreland	485,432	11,217	533	87	58,287	29,079	29,208
Wilts	865,092	51,667	2,250	176	254,221	125,728	128,493
Worcester	472,165	55,639	2,723	337	276,926	136,956	139,970
York (*East Riding*)	768,419	44,363	2,964	385	220,983	109,443	111,540
„ (*City*)	2,720	7,077	415	91	36,303	16,977	19,326
„ (*North Riding*)	1,350,121	44,446	2,343	224	215,214	106,710	108,504
„ (*West Riding*)	1,708,026	264,302	10,970	2,507	1,325,495	659,619	665,876
WALES.							
Anglesey	193,453	12,124	545	134	57,327	28,101	29,226
Brecon	460,158	12,221	731	74	61,474	31,314	30,160
Cardigan	443,387	14,978	544	70	70,796	32,961	37,835
Carmarthen	606,331	22,465	1,176	99	110,632	53,076	57,556
Carnarvon	370,273	18,005	590	132	87,870	42,978	44,892
Denbigh	386,052	19,124	812	136	92,583	46,708	45,875
Flint	184,905	14,041	798	80	68,156	34,452	33,704
Glamorgan	547,494	43,202	1,557	459	231,849	120,748	111,101
Merioneth	385,291	8,159	372	31	38,843	19,151	19,692
Montgomery	483,323	13,350	716	25	67,335	33,634	33,701
Pembroke	401,691	19,136	937	111	94,140	43,675	50,465
Radnor	272,128	4,614	217	28	24,716	12,693	12,023

Houses and Population enumerated (continued).

9	10	11	12	13	14	15	16	17
1861 (April 8th).						Increase or Decrease in the Number of Persons between 1851 and 1861.		Counties (Proper).
Houses.			Population.					
Inhabited.	Uninhabited.	Building.	Persons.	Males.	Females.	Increase.	Decrease.	
116,405	6,949	669	561,728	272,588	289,140	96,715	—	Warwick
11,809	603	76	60,809	30,665	30,144	2,522	—	Westmoreland
53,181	2,302	255	249,445	122,529	126,926	—	4,778	Wilts
62,893	3,791	355	307,601	150,989	156,612	30,675	—	Worcester
49,385	2,561	273	240,359	118,652	121,707	19,376	—	York (*East Riding*)
8,243	407	26	40,377	19,133	21,244	4,074	—	„ (*City*)
50,306	2,942	381	244,804	121,845	122,959	29,590	—	„ (*North Riding*)
316,061	18,555	1,998	1,507,511	740,696	766,815	182,016	—	„ (*West Riding*)
								WALES.
12,361	477	63	54,546	26,191	28,355	—	2,781	Anglesey
12,929	916	80	61,627	31,004	30,623	153	—	Brecon
15,731	572	109	72,255	33,105	39,150	1,459	—	Cardigan
23,106	915	129	111,757	53,166	58,591	1,125	—	Carmarthen
20,261	660	251	95,668	46,752	48,916	7,798	—	Carnarvon
21,386	618	259	100,862	51,027	59,835	8,279	—	Denbigh
15,146	781	137	69,870	34,744	35,126	1,714	—	Flint
59,356	3,721	736	317,751	163,271	154,480	85,902	—	Glamorgan
8,499	357	45	38,888	19,085	19,803	45	—	Merioneth
13,518	413	115	67,075	33,843	33,232	—	260	Montgomery
19,416	837	202	96,093	45,683	50,410	1,953	—	Pembroke
4,706	162	22	25,403	13,144	12,259	687	—	Radnor

Table VI.—*England and Wales. Houses and Population in Superintendent Registrars' Districts on March* 31*st,* 1851, *and on April* 8*th,* 1861.

[Five only out of the eleven divisions are here inserted, viz., *London, South Midland, Northern, North Western,* and *York.* These five will present a fair average of the whole.—Ed. *A. M.*]

No.	Superintendant Registrar's District.	Inhabited Houses.		Population Enumerated.		Increase or Decrease in the Number of Persons between 1851 and 1861.		Excess of Registered Births over Registered Deaths in the Ten Years, 1851-60.
		1851.	1861.	1851.	1861.	Increase.	Decrease.	
	1. LONDON DIVISION.							
	Middlesex (*part of*).							
1	Kensington (a)	17,151	25,854	120,004	186,463	66,459	—	14,209
2	Chelsea	7,591	8,318	56,538	63,423	6,885	—	4,302
3	St. George, Hanover Square	8,792	10,421	73,230	87,747	14,517	—	3,969
4	Westminster	6,642	6,880	65,609	67,676	2,067	—	4,259
5	St. Martin-in-the-Fields	2,307	2,283	24,640	22,636	—	2,004	457
6	St. James, Westminster	3,399	3,331	36,406	35,324	—	1,082	1,676
7	Marylebone	15,826	16,370	157,696	161,609	3,913	—	11,039
8	Hampstead	1,719	2,653	11,986	19,104	7,118	—	1,178
9	Pancras	18,584	21,928	166,956	198,882	31,296	—	19,649
10	Islington	13,528	20,676	95,329	155,291	59,962	—	15,881
11	Hackney	9,818	13,412	58,429	83,295	24,866	—	8,338
12	St. Giles	4,700	4,662	54,214	53,981	—	233	3,569
13	Strand (b)	3,949	3,815	44,417	42,956	—	1,461	2,516
14	Holborn (c)	4,311	4,125	46,621	44,861	—	1,760	2,028
15	Clerkenwell	7,224	7,086	64,778	65,632	854	—	8,236
16	St. Luke	6,349	6,368	54,055	56,997	2,942	—	11,610
17	East London	4,739	4,495	44,406	40,673	—	3,733	4,582
18	West London	2,670	2,616	28,833	27,144	—	1,689	— 4,647
19	London, City (d)	7,297	6,367	55,932	45,550	—	10,382	2,861
20	Shoreditch	15,337	17,231	109,257	129,339	20,082	—	18,402
21	Bethnal Green	13,298	14,812	90,193	104,905	14,712	—	15,963
22	Whitechapel	8,812	8,667	79,759	78,963	—	796	3,626
23	St. George-in-the-East	6,146	6,187	48,376	48,878	502	—	5,291
24a	Stepney	7,392	7,465	54,173	56,567	2,394	—	5,003
24b	Mile End Old Town	8,867	10,768	56,602	73,064	16,462	—	9,473
25	Poplar	6,831	11,163	47,162	79,182	32,020	—	9,466

Note.—The Superintendent Registrars' Districts are *generally* co-extensive with Poor Law Unions or Single Parishes under the Poor Law Amendment Act. Where this is not the case, the Districts are thus distinguished:—* *two or more* Unions or Single Parishes with Boards of Guardians; X Poor Law Unions with Places, not under the Poor Law Amendment Act, annexed for Registration purposes; † Single Parishes or Incorporations of Parishes under *Local Acts;* § Gilbert's Incorporations, or Parishes still under the Act of 43rd Elizabeth. The population of Unions, where stated in the notes, refers to the Census of 1861.

(a) The Superintendent Registrar's District of Kensington is composed of Kensington Parish (pop. 70,614 in 1861), the Fulham Union (pop. 40,042 in 1861), and Paddington Parish (pop. 75,807 in 1861).

(b) Strand Union and the Middle Temple.

(c) The Holborn Union, Gray's Inn, and the Charterhouse.

(d) West London Union and the Inner Temple.

TABLE VI.—*England and Wales. Houses and Population* (continued).

No.	Superintendent Registrar's District.	Inhabited Houses. 1851.	Inhabited Houses. 1861.	Population Enumerated. 1851.	Population Enumerated. 1861.	Increase or Decrease in the Number of Persons between 1851 and 1861. Increase.	Decrease.	Excess of Registered Births over Registered Deaths in the Ten Years, 1851–60.
	SURREY (*part of*).							
26	St. Saviour, Southwark	4,600	4,495	35,731	36,026	295	—	2,229
27	St. Olave, Southwark	2,360	2,214	19,375	19,053	—	322	— 5,447
28	Bermondsey	7,007	8,211	48,128	58,355	10,227	—	7,903
29	St. George, Southwark	6,992	7,234	51,824	55,509	3,685	—	5,337
30	Newington	10,458	12,815	64,816	82,157	17,341	—	10,015
31	Lambeth	20,447	23,001	139,325	162,008	22,683	—	19,884
32	Wandsworth	8,276	11,136	50,764	70,381	19,617	—	5,835
33	Camberwell	9,412	12,122	54,667	71,489	16,822	—	6,348
34	Rotherhithe	2,792	3,529	17,805	24,500	6,695	—	2,422
	KENT (*part of*).							
35	Greenwich	14,383	17,826	99,365	127,662	28,297	—	9,312
36	Lewisham	5,927	9,701	34,835	65,752	30,917	—	7,215
	3. SOUTH MIDLAND DIVISION.							
	6. MIDDLESEX (*extra metropolitan*).							
132	Staines	2,796	3,160	13,973	15,988	2,015	—	1,458
133	Uxbridge	3,524	4,018	19,475	23,139	3,664	—	2,121
134	Brentford	7,820	9,397	41,325	50,406	9,171	—	4,407
135	Hendon	2,811	3,384	15,916	19,207	3,291	—	1,802
136	Barnet	2,706	3,220	14,619	19,124	4,505	—	— 55
137	Edmonton	8,369	10,865	45,298	59,325	14,027	—	5,919
	7. HERTFORDSHIRE.							
138	Ware	3,259	3,341	16,482	16,516	34	—	2,029
139	Bishop Stortford	4,098	4,187	20,356	20,230	—	126	2,742
140	Royston * (a)	5,192	5,215	26,355	25,012	1,343	—	3,551
141	Hitchin	4,790	5,147	24,729	25,603	874	—	3,526
142	Hertford	2,762	2,912	15,090	15,301	211	—	1,549
143	Hatfield* (b)	1,569	1,676	8,499	8,400	—	99	1,062
144	St. Alban's	3,519	3,759	18,004	18,926	922	—	2,090
145	Watford	3,730	4,171	18,800	20,354	1,554	—	2,250
146	Hemel Hempstead	2,576	2,849	13,120	13,992	802	—	1,834
147	Berkhampstead	2,424	2,727	12,527	13,209	682	—	1,535
	8. BUCKINGHAMSHIRE.							
148	Amersham	3,774	3,820	18,637	18,238	—	399	2,112
149	Eton	3,942	4,362	21,490	22,356	866	—	2,180
150	Wycombe	6,630	7,198	33,562	35,139	1,576	—	4,663
151	Aylesbury	4,881	5,068	23,071	23,598	527	—	2,756
152	Winslow	1,969	2,016	9,376	9,207	—	169	1,169
153	Newport Pagnell	4,909	5,269	23,109	24,841	1,732	—	3,229
154	Buckingham	3,076	3,059	14,410	13,755	—	655	1,371

(a) Two Unions, viz., the Royston Union (pop. 18,623) and the Buntingford Union (pop. 6,389).
(b) Two Unions, viz., the Hatfield Union (pop. 6,389) and the Welwyn Union (pop. 2,211).

Table VI.—*England and Wales. Houses and Population* (continued).

No.	Superintendent Registrar's District.	Inhabited Houses. 1851.	Inhabited Houses. 1861.	Population Enumerated. 1851.	Population Enumerated. 1861.	Increase or Decrease in the Number of Persons between 1851 and 1861. Increase.	Decrease.	Excess of Registered Births over Registered Deaths in the Ten Years, 1851-60.
	9. Oxfordshire.							
155	Henley	3,579	3,722	17,895	18,209	314	—	2,179
156	Thame	3,261	3,277	15,640	15,306	—	334	1,910
157	Headington	3,067	3,494	15,771	17,182	1,411	—	1,517
158	Oxford†	3,528	3,770	20,172	20,038	—	134	2,111
159	Bicester	3,277	3,879	15,562	15,553	—	9	2,097
160	Woodstock	2,960	3,074	14,453	14,236	—	217	1,564
161	Witney	4,897	5,117	23,558	23,231	—	327	2,527
162	Chipping Norton	3,506	3,633	17,427	17,307	—	120	2,121
163	Banbury	6,384	6,739	29,769	30,161	392	—	3,658
	10. Northamptonshire.							
164	Brackley	2,961	3,021	13,747	13,464	—	283	1,588
165	Towcester	2,847	2,957	12,806	13,003	197	—	1,434
166	Potterspury	2,131	2,308	10,663	11,621	958	—	1,312
167	Hardingstone	1,883	2,146	9,157	9,928	771	—	1,464
168	Northampton	6,433	7,960	33,857	41,159	7,302	—	4,553
169	Daventry	4,527	4,679	21,926	20,600	—	1,326	2,493
170	Brixworth	3,229	3,328	14,771	15,358	587	—	1,632
171	Wellingborough	4,476	5,203	21,367	24,234	2,867	—	3,229
172	Kettering	3,770	4,081	18,097	19,086	989	—	2,887
173	Thrapston	2,711	3,071	12,841	14,064	1,223	—	2,063
174	Oundle	3,167	3,221	15,655	15,462	—	193	2,232
175	Peterborough	5,793	6,877	28,957	33,164	4,207	—	5,274
	11. Huntingdonshire.							
176	Huntingdon	4,218	4,327	20,900	20,516	—	384	2,990
177	St. Ives	4,316	4,278	20,594	19,649	—	945	2,641
178	St. Neots	3,930	4,092	18,825	18,962	137	—	2,719
	12. Bedfordshire.							
179	Bedford	7,189	7,926	35,523	38,063	2,540	—	4,623
180	Biggleswade	4,609	5,022	23,436	25,389	1,953	—	3,763
181	Ampthill	3,221	3,521	16,542	16,970	428	—	2,177
182	Woburn	2,435	2,491	12,075	11,682	—	393	1,348
183	Leighton Buzzard	3,407	3,566	17,142	17,641	499	—	2,540
184	Luton	4,848	5,851	25,087	30,705	5,618	—	4,214
	13. Cambridgeshire.							
185	Caxton	2,132	2,202	11,065	10,966	—	99	1,861
186	Chesterton	5,209	5,465	25,170	25,082	—	88	3,397
187	Cambridge	5,194	5,411	27,815	26,351	—	1,464	2,204
188	Linton	2,857	2,895	14,148	13,509	—	639	1,952
189	Newmarket	6,104	6,123	30,655	28,776	—	1,879	3,269
190	Ely	4,641	4,709	22,896	21,805	—	1,091	3,135
191	North Witchford	3,331	3,253	16,243	14,787	—	1,456	2,415
192	Whittlesey	1,678	1,594	7,687	6,966	—	721	1,115
193	Wisbeach	7,590	7,528	36,215	33,304	—	2,911	4,234

† Under Local Acts.

TABLE VI.—*England and Wales. Houses and Population* (continued).

No.	Superintendent Registrar's District.	Inhabited Houses. 1851.	Inhabited Houses. 1861.	Population Enumerated. 1851.	Population Enumerated. 1861.	Increase or Decrease in the Number of Persons between 1851 and 1861. Increase.	Decrease.	Excess of Registered Births over Registered Deaths in the Ten Years, 1851-60.
	4. EASTERN DIVISION.							
	14. ESSEX.							
194	West Ham	6,003	9,809	34,395	59,261	24,866	—	6,089
195	Epping	3,086	3,385	15,631	16,544	913	—	1,726
196	Ongar	2,281	2,262	11,855	11,314	—	541	1,466
197	Romford	4,758	5,419	24,607	26,920	2,313	—	3,722
198	Orsett	1,878	2,128	10,642	11,529	887	—	1,499
199	Billericay	2,526	2,763	13,787	15,013	1,226	—	1,218
200	Chelmsford	6,566	6,921	32,272	32,793	521	—	3,497
201	Rochford	3,069	3,382	15,838	18,270	2,432	—	1,910
202	Maldon	4,471	4,579	22,137	22,573	436	—	1,999
203	Tendring	5,339	5,784	27,710	27,094	—	616	2,926
204	Colchester	4,145	4,459	19,443	23,815	4,372	—	1,893
205	Lexden	4,628	4,966	21,666	22,922	1,256	—	3,593
206	Witham	3,306	3,452	16,099	16,324	225	—	1,931
207	Halstead	4,040	4,137	19,273	18,482	—	791	2,105
208	Braintree	3,770	3,841	17,561	17,169	—	392	1,988
209	Dunmow	4,249	4,373	20,498	19,759	—	739	2,625
210	Saffron Walden	4,260	4,294	20,716	19,721	—	995	2,604
	15. SUFFOLK.							
211	Risbridge	3,724	3,708	18,125	17,432	—	693	2,613
212	Sudbury	6,553	6,939	30,814	31,414	600	—	3,863
213	Cosford	3,790	3,884	18,107	17,374	—	733	2,154
214	Thingoe	3,827	3,864	19,014	18,221	—	793	2,540
215	Bury St. Edmund's†	2,752	2,847	13,900	13,316	—	584	922
216	Mildenhall	2,093	2,051	10,354	9,592	—	762	1,388
217	Stow	4,320	4,474	21,110	20,917	—	193	2,751
218	Hartismere	3,670	3,693	19,028	17,664	—	1,364	2,354
219	Hoxne	3,231	3,177	15,900	14,695	—	1,205	1,953
220	Bosmere	3,581	3,560	17,219	16,173	—	1,046	2,100
221	Samford	2,560	2,766	12,493	12,729	236	—	1,644
222	Ipswich	6,949	8,266	32,759	37,880	5,121	—	4,046
223	Woodbridge	5,044	5,068	23,776	22,748	—	1,028	2,339
224	Plomesgate	4,377	4,599	21,477	20,719	—	758	2,234
225	Blything	5,870	5,878	27,883	26,859	—	1,024	3,639
226	Wangford	2,922	2,977	14,014	13,620	—	394	1,408
227	Mutford†	4,061	5,193	20,163	24,069	3,906	—	3,020
	16. NORFOLK.							
228	Yarmouth	6,006	6,856	26,880	30,318	3,438	—	2,186
229	Flegg†	1,822	1,958	8,497	8,630	133	—	1,356
230	Tunstead†	3,402	3,346	15,614	14,515	—	1,099	1,480
231	Erpingham X (a)	4,698	4,895	21,722	20,875	—	847	2,100
232	Aylsham	4,323	4,300	20,007	19,050	—	957	2,142
233	St. Faith's	2,459	2,538	11,890	11,752	—	138	1,536

† Under Local Acts.
(a) The Erpingham Union (pop. 20,580 in 1861) and the Brinton Gilbert's Incorporation.

TABLE VI.—*England and Wales. Houses and Population* (continued).

No.	Superintendent Registrar's District.	Inhabited Houses. 1851.	Inhabited Houses. 1861.	Population Enumerated. 1851.	Population Enumerated. 1861.	Increase or Decrease in the Number of Persons between 1851 and 1861. Increase.	Decrease.	Excess of Registered Births over Registered Deaths in the Ten Years, 1851-1860.
	16. NORFOLK—*contd.*							
234	Norwich†	14,988	17,012	68,195	74,414	6,219	—	5,751
235	Forehoe†	2,909	2,895	13,565	12,818	—	747	1,228
236	Henstead	2,375	2,432	11,545	11,290	—	255	1,321
237	Blofield	2,309	2,358	11,574	11,521	—	53	1,081
238	Loddon	3,138	3,137	15,095	14,242	—	853	1,878
239	Depwade	5,356	5,602	26,395	25,249	—	1,146	3,194
240	Guiltcross	2,626	2,580	12,744	11,541	—	1,203	1,308
241	Wayland	2,528	2,555	12,141	11,558	—	583	1,449
242	Mitford	6,147	6,149	29,389	28,018	—	1,371	3,131
243	Walsingham	4,610	4,735	21,883	21,115	—	768	2,327
244	Docking	3,716	3,794	18,148	17,579	—	569	1,968
245	Freebridge Lynn	2,689	2,833	13,557	13,474	—	83	2,019
246	King's Lynn	4,028	3,768	20,530	16,602	—	3,928	1,499
247	Downham	4,317	4,337	20,985	20,260	—	725	2,977
248	Swaffham	2,843	2,924	14,320	13,745	—	575	1,821
249	Thetford	3,855	4,048	19,040	18,711	—	329	2,215
	8. NORTH-WESTERN DIVISION.							
	33. CHESHIRE.							
452	Stockport	17,323	19,167	90,208	94,361	4,152	—	9,664
453	Macclesfield	12,845	13,386	63,327	61,517	—	1,810	5,561
454	Altrincham	6,363	7,784	34,043	40,515	6,473	—	4,794
455	Runcorn	4,832	5,235	25,797	26,129	332	—	3,805
456	Northwich	6,221	6,938	31,202	33,331	2,129	—	5,015
457	Congleton	5,893	7,012	30,512	34,329	3,817	—	3,551
458	Nantwich	6,876	8,079	35,941	40,954	5,013	—	5,776
459	Great Boughton*† ([a])	9,889	11,248	52,950	58,503	5,553	—	4,437
460	Wirrall	9,109	12,227	57,157	79,826	22,669	—	11,230
	34. LANCASHIRE.							
461	Liverpool†	35,293	37,045	258,236	269,733	11,497	—	2,230
462	West Derby* ([b])	25,031	37,512	153,279	225,595	72,316	—	24,596
463	Prescot	9,323	13,120	56,074	73,112	17,038	—	11,308
464	Ormskirk	6,737	8,364	38,307	46,250	7,943	—	6,219
465	Wigan	13,965	17,422	77,539	94,559	17,020	—	13,227
466	Warrington	6,647	8,229	36,164	43,788	7,624	—	6,798
467	Leigh	6,015	7,618	32,734	37,697	4,963	—	4,583
468	Bolton	20,240	25,155	114,712	130,270	15,558	—	17,789

† Under Local Acts.

([a]) The Superintendent Registrar's District of Great Boughton is composed of—

	Population, 1861.
Chester City (Local Act)	30,116
Great Boughton Union	18,795
Hardwarden Union	9,592

([b]) The West Derby Union (pop. 156,327 in 1861) and Toxteth Park (pop. 69,268) a single parish under the Poor Law Amendment Act.

TABLE VI.—*England and Wales. Houses and Population* (continued).

No.	Superintendent Registrar's District.	Inhabited Houses. 1851.	Inhabited Houses. 1861.	Population Enumerated. 1851.	Population Enumerated. 1861.	Increase or Decrease in the Number of Persons between 1851 and 1861. Increase.	Decrease.	Excess of Registered Births over Registered Deaths in the Ten Years, 1851-60.
	34. LANCASHIRE—*contd.*							
469	Bury	16,727	19,803	88,815	101,142	12,327	—	12,336
470	Barton-upon-Irwell	5,737	7,465	31,585	39,050	7,465	—	4,051
471	Chorlton	23,159	32,879	123,841	169,573	45,732	—	21,601
472	Salford	15,769	19,831	87,523	105,334	17,811	—	11,902
473	Manchester	36,701	43,003	228,433	243,615	15,182	—	16,929
474	Ashton	21,569	26,527	119,199	134,761	15,562	—	12,792
475	Oldham	16,485	21,951	86,788	111,267	24,479	—	13,046
476	Rochdale	14,200	18,407	72,515	91,758	19,243	—	9,763
477	Haslingden	9,489	13,402	50,424	69,782	19,358	—	8,615
478	Burnley	12,039	14,532	63,868	75,588	11,720	—	9,145
479	Clitheroe	4,238	4,146	22,368	20,476	—	1,892	2,029
480	Blackburn	15,916	21,888	90,738	119,937	29,199	—	14,861
481	Chorley	6,723	7,869	37,701	41,679	3,978	—	5,773
482	Preston	15,913	20,071	96,545	110,488	13,943	—	11,046
483	Fylde	3,930	4,581	22,002	25,681	3,679	—	2,977
484	Garstang	2,364	2,378	12,695	12,424	—	271	1,741
485	Lancaster X ([a])	6,272	6,704	34,660	35,299	639	—	3,664
486	Ulverstone	5,676	6,844	30,556	35,734	5,178	—	5,168
	9. YORK DIVISION.							
	35. WEST RIDING.							
487	Sedbergh	917	915	4,574	4,396	—	178	536
488	Settle	2,774	2,608	13,762	12,529	—	1,233	1,408
489	Skipton X ([b])	5,719	6,047	28,766	28,761	—	5	3,566
490	Pateley Bridge	1,928	1,992	9,334	9,534	200	—	1,105
491	Ripon	3,471	3,485	16,041	15,742	—	299	1,939
492*a*	Great Ouseburn	2,579	2,509	12,167	11,532	—	635	984
492*b*	Knaresborough	3,448	3,741	15,473	17,176	1,703	—	1,861
492*c*	Wetherby §	1,055	1,092	5,129	5,123	—	6	411
493	Otley §	5,588	6,069	28,541	29,508	967	—	3,304
494	Keighley X ([c])	8,638	9,597	45,903	45,681	—	222	5,548
495	Todmorden	5,940	6,436	29,727	31,105	1,378	—	3,114
496	Saddleworth	3,367	3,808	17,799	18,630	831	—	2,037
497	Huddersfield	23,468	26,658	128,860	131,334	7,474	—	19,671
498	Halifax	23,626	27,016	120,958	128,667	7,709	—	15,917
499	Bradford X ([d])	34,439	41,860	181,964	196,463	14,499	—	26,242
500	Hunslet §	18,776	23,895	88,679	109,949	21,270	—	15,331
501	Leeds	21,061	24,999	101,343	117,553	16,210	—	11,079

§ Parishes, &c. in Gilbert's Incorporations or under the Act of 43rd Elizabeth.

([a]) The Lancaster Union (pop. 24,019 in 1861), Caton Gilbert's Incorporation, and Parishes under 43rd Elizabeth.

([b]) The Skipton Union (pop. 28,398) and two townships not under the Poor Law Amendment Act.

([c]) The Keighley Union (pop. 43,112) and a township under 43rd Elizabeth.

([d]) The Bradford Union (pop. 106,218), the North Bierley Union (pop. 85,768), and a part of a Gilbert's Incorporation.

TABLE VI.—*England and Wales. Houses and Population* (continued.)

No.	Superintendent Registrar's District.	Inhabited Houses. 1851.	Inhabited Houses. 1861.	Population Enumerated. 1851.	Population Enumerated. 1861.	Increase or Decrease in the Number of Persons between 1851 and 1861. Increase.	Decrease.	Excess of Registered Births over Registered Deaths in the Ten Years, 1851-60.
	35. WEST RIDING—*contd.*							
502	Dewsbury	14,351	19,381	71,768	92,873	21,105	—	13,813
503	Wakefield X (a)	9,874	10,815	48,956	53,001	4,045	—	6,119
504a	Pontefract §	6,353	7,475	29,937	34,752	4,815	—	5,028
504b	Hemsworth	1,658	1,657	8,158	7,793	—	365	1,024
505	Barnsley	6,777	9,102	34,980	45,790	10,810	—	7,209
506	Wortley X (b)	6,255	7,577	32,012	38,509	6,497	—	6,598
507	Ecclesall Bierlow	7,587	13,009	37,914	63,618	25,704	—	8,651
508	Sheffield	20,785	26,658	103,626	128,929	25,303	—	16,281
509	Rotherham	6,686	9,055	33,082	44,330	11,248	—	6,826
510	Doncaster	7,302	8,625	34,675	39,341	4,666	—	5,020
511	Thorne	3,518	3,650	15,886	16,010	124	—	2,203
512	Goole	2,892	3,332	13,686	15,156	1,470	—	1,908
513	Selby X (c)	3,370	3,547	15,672	15,985	313	—	1,858
514	Tadcaster §	4,093	4,223	19,710	19,919	209	—	2,317
	36. EAST RIDING (*with York*).							
515	York X (d)	10,469	11,941	54,324	59,967	5,643	—	5,306
516	Pocklington	3,250	3,420	16,098	16,710	612	—	2,128
517	Howden	3,060	3,159	14,436	15,076	640	—	2,419
518	Beverley	4,167	4,674	20,040	21,029	989	—	2,623
519	Sculcoates	9,187	11,136	44,719	51,942	7,223	—	6,236
520	Hull †	9,733	10,882	50,670	56,889	6,219	—	4,161
521	Patrington	1,872	2,010	9,407	9,680	273	—	1,316
522	Skirlaugh	1,868	1,952	9,279	9,653	374	—	1,297
523	Driffield	3,684	3,953	18,265	19,223	958	—	2,923
524	Bridlington	2,926	3,082	14,322	14,371	49	—	1,805
	37. NORTH RIDING.							
525	Scarborough	5,905	6,401	24,615	30,424	5,809	—	3,410
526	Malton	4,546	4,734	23,128	23,482	354	—	3,264
527	Easingwold	2,075	2,098	10,211	10,148	—	63	1,375
528	Thirsk	2,682	2,695	12,760	12,299	—	461	1,525
529	Helmsley X (e)	2,366	2,436	11,734	11,832	98	—	1,376
530	Pickering	2,083	2,197	9,978	10,547	569	—	1,373
531	Whitby	4,490	4,915	21,592	23,634	2,042	—	3,212
532	Guisbrough	2,544	4,367	12,202	22,125	9,923	—	3,173
533	Stokesley	2,984	2,226	9,387	10,381	994	—	1,309
534	Northallerton	2,650	2,661	12,460	12,174	—	286	1,399
535	Bedale	1,914	1,945	8,980	8,650	—	330	992
536	Leyburn	2,082	2,103	10,057	10,104	47	—	1,319
537	Askrigg ‖	1,248	1,229	5,635	5,649	14	—	757
538	Reeth	1,399	1,286	6,820	6,195	—	625	1,047
539	Richmond	2,894	2,842	13,846	13,456	—	390	1,464

† Under Local Act.
§ Parishes, &c., in Gilbert's Incorporations or under the Act of 43rd Elizabeth.
‖ Parishes, &c., not under the Poor Law Amendment Act.
(a) The Wakefield Union (pop. 51,092) and parts of Gilbert's Incorporations.
(b) Two Unions, viz., Wortley (pop. 24,091) and Penistone (pop. 14,418).
(c) The Selby Union (pop. 14,902) and parts of Gilbert's Incorporations.
(d) The York Union (pop. 59,157) and parishes not under Poor Law Amendment Act.
(e) Two Unions, viz., Helmsley (pop. 6,093) and Kirkby Moorside (pop. 5,739).

TABLE VII.—*England and Wales. Population in* 1851 *and* 1861, *ascertained Increase or Decrease in the interval, and Natural Increase or Excess of Registered Births over Deaths in the Ten Years*, 1851-60.

1	2	3	4	5
DIVISIONS.	Population Enumerated.		*Ascertained Increase* in the interval between 1851 and 1861.	*Natural Increase* or Excess of Registered *Births* over Registered *Deaths*, in the Ten Years, 1851-60.
	1851.	1861.		
ENGLAND AND WALES	17,927,609	20,061,725	2,134,116	2,260,576
DIVISIONS.				
I. LONDON (within the limits of the Metropolis Local Government Act)	2,362,236	2,803,034	440,798	253,989
II. SOUTH EASTERN (Registration Districts chiefly in *Surrey* and *Kent* extra-metropolitan, *Sussex*, *Hants*, *Berks*)	1,628,416	1,846,876	218,460	196,992
III. SOUTH MIDLAND (Registration Districts chiefly in *Middlesex* extra-metropolitan, *Herts*, *Bucks*, *Oxford*, *Northampton*, *Hunts*, *Beds*, *Cambridge*)	1,234,332	1,295,375	61,043	*155,742
IV. EASTERN (Registration Districts chiefly in *Essex*, *Suffolk*, *Norfolk*)	1,113,982	1,142,202	28,220	*129,726
V. SOUTH-WESTERN (Registration Districts chiefly in *Wilts*, *Dorset*, *Devon*, *Cornwall*, *Somerset*)	1,803,261	1,835,551	32,290	*200,673
VI. WEST MIDLAND (Registration Districts chiefly in *Gloucester*, *Hereford*, *Salop*, *Stafford*, *Worcester*, *Warwick*)	2,136,573	2,436,137	299,564	298,980
VII. NORTH MIDLAND (Registration Districts chiefly in *Leicester*, *Rutland*, *Lincoln*, *Nottingham*, *Derby*)	1,215,501	1,288,718	73,217	*161,763
VIII. NORTH-WESTERN (Registration Districts chiefly in *Cheshire*, *Lancashire*) .	2,488,438	2,934,722	446,284	308,022

Table VII.—*England and Wales. Population in* 1851 *and* 1861 (continued).

1 Divisions.	2 Population Enumerated. 1851.	3 1861.	4 *Ascertained Increase* in the interval between 1851 and 1861.	5 *Natural Increase* or Excess of Registered *Births* over Registered *Deaths*, in the Ten Years, 1851-60.
IX. York (Registration Districts chiefly in *Yorkshire*) . . .	1,789,047	2,015,329	226,282	*256,117
X. Northern (Registration Districts chiefly in *Durham, Northumberland, Cumberland, Westmoreland*)	969,126	1,151,281	182,155	152,694
XI. Welsh (Registration Districts chiefly in *Monmouthshire* and Wales) . .	1,186,697	1,312,500	125,803	*145,878

Note.—In the cases marked (*) the ascertained (actual) increase (col. 4) falls *short* of the natural (presumed) increase as shown by excess of Births over Deaths.

The following is a summary of *Occupations* as ascertained, 1851.

Registration Divisions and Union Counties (England and Wales).	Number of Persons Aged 20 Years and upwards, 1851.	Ratio per Cent. of Persons Aged 20 Years and upwards, occupied (in 1851) in Mechanical Arts, Trade, and Domestic Service.	 Agriculture.	 Manufactures.	 Mining and Mineral Works.
		Per Cent.	Per Cent.	Per Cent.	Per Cent.
I. The Metropolis	1,394,963	47·6	1·1	6·0	3·5
II. South-Eastern	887,134	30·7	20·8	2·5	2·4
III. South Midland	660,775	28·3	25·4	7·1	2·4
IV. Eastern	603,720	27·4	26·5	4·0	2·3
V. South-Western	978,024	28·6	23·3	4·6	5·6
VI. West Midland	1,160,387	29·1	15·5	5·2	12·6
VII. North Midland	654,679	31·8	21·7	6·4	5·3
VIII. North-Western	1,351,830	29·8	8·3	21·5	5·4
IX. York	961,945	25·2	14·3	17·5	7·3
X. Northern	521,460	27·7	16·1	4·2	12·4
XI. Welsh	641,680	21·8	25·7	2·5	12·4
England and Wales . .	9,816,597	31·0	16·1	8·4	6·3

TABLE VIII.—*Houses and Population of the Principal Cities and Boroughs,* 1851 *and* 1861.

*** "These Statements of the Number of Houses and Population in the principal Cities and Boroughs have been furnished generally by the respective Superintendent Registrars, and refer to the *Parliamentary or Municipal Limits*, or to both where co-extensive, as denoted by the letters placed after the names of places, viz., M., Municipal limits; P., Parliamentary limits; M. and P., Municipal and Parliamentary limits (co-extensive).

"The numbers of seamen and others on board vessels in the ports, as ascertained by the officers of Her Majesty's Customs, have been added to the general population. Several Boroughs, with respect to which the information could not be obtained, are omitted."

1	2	3	4	5	6
City or Borough.	Inhabited Houses.		Population.		
	1851.	1861.	1851.	1861.	
London, within the limits of the Metropolis Local Government Act*	305,933	362,890	2,362,236	2,803,034	—
Abingdon .... M. & P.	1,244	1,187	5,954	—	5,691
Andover .... M.	1,040	1,059	5,187	5,221	—
Andover .... P.	1,079	1,103	5,395	5,430	—
Arundel .... M. & P.	552	529	2,748	—	2,488
Ashburton .... P.	622	574	3,432	—	3,062
Ashton-under-Lyne .... M.	5,501	6,665	30,676	34,894	—
Ashton-under-Lyne .... P.	5,346	6,478	29,791	33,925	—
Banbury .... M.	769	790	4,026	4,055	—
Banbury .... P.	1,721	2,067	8,715	10,194	—
Bangor .... P.	1,228	1,336	6,338	6,795	—
Barnstaple .... M. & P.	2,116	2,187	11,371	—	10,738
Basingstoke .... M.	892	938	4,263	4,664	—
Bath .... M. & P.	7,744	8,021	54,240	—	52,528
Beccles .... M.	954	985	4,398	—	4,266
Bedford .... M. & P.	2,307	2,754	11,693	13,412	—
Berwick-on-Tweed .... M. & P.	2,028	1,872	15,094	—	13,254
Beverley .... M.	1,934	2,178	8,915	9,654	—
Beverley .... P.	2,183	2,423	10,058	10,901	—
Bewdley .... M.	718	691	3,124	—	2,905
Bewdley .... P.	1,582	1,516	7,318	—	6,786
Bideford .... M.	1,101	1,211	5,775	5,851	—
Birmingham .... M. & P.	45,844	59,090	232,841	295,955	—
Blackburn .... M. & P.	7,919	11,314	46,536	63,125	—
Bodmin .... M.	722	793	4,327	4,466	—
Bodmin .... P.	1,103	1,189	6,337	6,381	—

Note.—Col. 6 contains the cases in which in 1861 the population is *less* than in 1851.

* For the Metropolitan *Parliamentary* Boroughs, *see* Alphabetical List.

TABLE VIII.—*Houses and Population* (continued).

1		2	3	4	5	6
City or Borough.		Inhabited Houses.		Population.		
		1851.	1861.	1851.	1861.	
Bolton	M. & P.	10,394	13,348	61,171	70,396	—
Boston	M.	2,992	3,266	14,733	—	13,995
	P.	3,622	3,898	17,518	17,885	—
Bradford	M. & P.	19,002	22,537	103,778	106,218	—
Brecknock	M.	1,147	1,134	5,673	—	5,234
	P.	1,236	1,206	6,070	—	5,517
Brydgnorth	M.	1,227	1,629	6,172	6,569	—
	P.	1,516	1,891	7,610	7,892	—
Bridgwater	M. & P.	1,911	2,124	10,317	11,361	—
Bridport	M. & P.	1,468	1,570	7,566	7,672	—
Brighton	M.	*	12,708	*	77,693	—
	P.	10,843	13,946	69,673	87,311	—
Bristol	M. & P.	20,873	23,578	137,328	154,093	—
Buckingham	M.	809	827	4,020	—	3,847
	P.	1,717	1,716	8,069	—	7,625
Bury	P.	5,825	7,241	31,262	37,564	—
Bury St. Edmund's	M. & P.	2,752	2,847	13,900	—	13,316
Calne	M.	475	505	2,544	—	2,494
	P.	1,047	1,103	5,195	—	5,151
Cambridge	M. & P.	5,194	5,411	27,815	—	26,351
Canterbury	M. & P.	3,654	3,919	18,398	21,323	—
Cardiff	M.	2,565	4,666	18,351	32,421	—
Carlisle	M. & P.	3,956	4,878	26,310	29,436	—
Carmarthen	M. & P.	1,800	1,763	10,524	—	9,992
Carnarvon	M. & P.	1,723	1,820	8,674	—	8,530
Chatham	P.	4,337	5,227	28,424	36,177	—
Cheltenham	P.	6,356	7,016	35,051	39,590	—
Chester	M. & P.	5,173	5,980	27,766	31,101	—
Chesterfield	M.	1,455	1,928	7,101	9,835	—
Chichester	M. & P.	1,653	1,597	8,662	—	8,040
Chippenham	M.	309	300	1,707	—	1,603
	P.	1,139	1,345	6,283	7,075	—
Chipping Wycombe	M.	690	825	3,588	4,222	—
	P.	1,441	1,652	7,179	8,375	—
Christchurch	P.	1,543	1,837	7,475	9,386	—
Cirencester	P.	1,211	1,300	6,096	6,334	—
Clitheroe	M.	1,371	1,433	7,244	—	7,000
	P.	2,192	2,247	11,480	—	10,864
Cockermouth	P.	1,306	1,546	7,275	—	7,056
Colchester	M. & P.	4,145	4,459	19,443	23,815	—
Congleton	M.	2,146	2,631	10,520	12,338	—
Coventry	M.	7,657	8,994	36,208	40,937	—
	P.	7,783	9,158	36,812	41,647	—
Dartmouth	M. & P.	799	822	4,508	—	4,443
Daventry	M.	889	892	4,430	—	4,124
Deal	M.	1,465	1,590	7,067	7,531	—

* Brighton had not received a Charter of Incorporation in 1851.

TABLE VIII.—*Houses and Population* (continued).

1		2	3	4	5	6
City or Borough.		Inhabited Houses.		Population.		
		1851.	1861.	1851.	1861.	
Denbigh	M.	1,215	1,261	5,498	5,946	—
Derby	M. & P.	8,199	9,014	40,908	43,091	—
Devizes	M. & P.	1,292	1,389	6,554	6,639	—
Devonport	M.	3,789	4,193	38,180	50,504	—
	P.	4,961	5,435	50,159	64,798	—
Doncaster	M.	2,583	3,595	12,052	16,430	—
Dorchester	M. & P.	960	1,028	6,394	6,823	—
Dover	M. & P.	3,747	4,087	22,244	24,970	—
Droitwich	M.	582	672	3,125	—	3,123
	P.	1,407	1,406	7,095	—	6,540
Dudley	P.	7,119	8,725	37,962	44,975	—
Durham	M. & P.	1,768	2,001	13,188	13,743	—
Evesham	M. & P.	918	991	4,605	4,680	—
Exeter	M.	5,109	5,404	32,818	33,737	—
	P.	6,499	6,885	40,688	41,791	—
Eye	M.	480	489	2,587	—	2,430
	P.	1,374	1,406	7,531	—	7,039
Falmouth	M.	600	669	4,953	5,706	—
Faversham	M.	895	1,125	4,595	5,891	—
Finsbury	P.	37,427	44,363	323,772	386,844	—
Folkestone	M.	1,149	1,475	6,726	8,528	—
Frome	P.	2,122	2,069	10,148	—	9,523
Gateshead	M. & P.	3,520	4,394	25,568	33,589	—
Gloucester	M. & P.	2,843	2,769	17,572	—	16,320
Grantham	M.	904	940	5,375	—	4,954
	P.	1,968	2,256	10,873	11,116	—
Gravesend	M.	2,722	3,074	16,633	18,776	—
Great Grimsby	M.	1,634	2,302	8,860	11,067	—
	P.	2,354	3,177	12,263	15,013	—
Great Marlow	P.	1,211	1,287	6,523	—	6,505
Great Yarmouth	M. & P.	6,886	7,836	30,879	34,803	—
Greenwich	P.	15,401	19,500	105,784	139,286	—
Guildford	M. & P.	1,176	1,464	6,740	8,032	—
Halifax	M. & P.	6,528	7,820	33,582	37,015	—
Hartlepool	M.	1,466	1,723	9,503	12,205	—
Harwich	M. & P.	751	812	4,451	5,062	—
Hastings	M.	2,471	3,327	16,966	23,098	—
	P.	2,477	3,328	17,011	23,103	—
Helston	M.	672	745	3,355	3,841	—
	P.	1,459	1,714	7,328	8,657	—
Hereford	M. & P.	2,426	No ret.	12,108	15,625	—
Hertford	M. & P.	1,150	1,230	6,605	6,769	—
Holyhead	P.	1,040	1,256	5,622	6,190	—
Honiton	M. & P.	692	714	3,427	—	3,301
Horsham	P.	1,081	1,267	5,947	6,747	—
Huddersfield	P.	5,739	6,933	30,880	34,874	—
Hull	M. & P.	16,634	20,581	84,690	98,994	—
Huntingdon	M.	725	740	3,882	—	3,816
	P.	1,244	1,284	6,219	6,254	—

Table VIII.—*Houses and Population* (continued).

1		2	3	4	5	6
City or Borough.		Inhabited Houses.		Population.		
		1851.	1861.	1851.	1861.	
Hythe	M.	486	588	2,857	2,998	—
	P.	2,261	2,841	13,164	21,372	—
Ipswich	M. & P.	6,979	8,284	32,914	37,949	—
Kendal	M. & P.	2,457	2,582	11,829	12,028	—
Kidderminster	M. & P.	3,656	3,349	18,462	—	15,398
King's Lynn	M. & P.	3,845	3,641	19,355	—	16,071
Kingston-on-Thames	M.	1,119	1,519	6,279	9,114	—
Knaresborough	P.	1,326	1,320	5,536	—	5,404
Lambeth	P.	39,154	45,252	251,345	298,032	—
Lancaster	M.	2,583	2,680	14,604	—	14,478
	P.	2,891	No ret.	16,168	—	15,996
Launceston	M.	562	624	3,397	—	2,773
	P.	1,051	1,040	6,005	—	5,139
Leeds	M. & P.	36,165	44,646	172,270	207,153	—
Leicester	M. & P.	12,805	14,680	60,584	68,052	—
Leominster	M. & P.	1,118	1,158	5,214	5,660	—
Lewes	P.	1,747	No ret.	9,533	9,709	—
Lichfield	M. & P.	1,412	1,471	7,012	—	6,872
Lincoln	M. & P.	3,450	4,332	17,536	20,995	—
Liskeard	M.	623	781	4,386	4,689	—
	P.	965	1,170	6,204	6,704	—
Liverpool	M. & P.	54,310	65,999	375,955	443,874	—
London, City	M. & P.	14,580	13,373	127,869	—	112,247
Louth	M.	2,209	2,401	10,467	10,568	—
Ludlow	M.	1,003	1,076	4,691	5,178	—
	P.	1,133	1,264	5,376	6,034	—
Lyme Regis	M.	522	507	2,661	—	2,413
	P.	708	687	3,516	—	3,206
Lymington	M.	487	440	2,651	—	2,416
	P.	1,029	1,025	5,282	—	5,152
Macclesfield	M. & P.	8,312	8,345	39,048	—	36,095
Maidstone	M.	3,667	4,111	20,740	22,984	—
	P.	3,676	4,119	20,801	23,026	—
Maldon	M.	902	1,014	4,558	4,798	—
	P.	1,179	1,329	5,888	6,274	—
Malmesbury	P.	1,420	1,425	6,998	—	6,883
Malton	P.	1,545	1,696	7,661	8,072	—
Manchester	M.	50,731	61,662	303,382	338,346	—
	P.	53,204	65,553	316,213	357,604	—
Marlborough	M.	608	644	3,908	—	3,684
	P.	781	820	5,135	—	4,893
Marylebone	P.	40,513	48,027	370,957	436,298	—
Merthyr Tydfil	P.	11,684	16,147	63,080	83,844	—
Morpeth	P.	1,467	2,325	10,012	13,796	—
Newark	M. & P.	2,370	2,569	11,330	11,562	—
Newbury	M.	1,362	1,337	6,574	—	6,161
Newcastle-under-Lyme	M. & P.	2,153	2,657	10,569	12,938	—
Newcastle-on-Tyne	M. & P.	10,441	14,222	87,784	109,291	—
Newport (I. Wight)	M. & P.	1,550	1,591	8,047	—	7,934

TABLE VIII.—*Houses and Population* (continued).

1	2	3	4	5	6
	Inhabited Houses.		Population.		
CITY OR BOROUGH.	1851.	1861.	1851.	1861.	
Newport (Monmouth) . . M. & P.	2,908	3,695	19,323	23,248	—
Northallerton . . . . P.	1,064	1,059	4,995	—	4,755
Northampton . . . . M. & P.	4,886	6,132	26,657	32,813	—
Norwich . . . . M. & P.	14,988	17,012	68,195	74,414	—
Nottingham . . . . M. & P.	11,549	15,307	57,407	74,531	—
Oldham . . . . M.	9,900	13,936	52,820	72,334	—
Oldham P.	13,658	18,463	72,357	94,337	—
Oswestry . . . . M.	995	1,129	4,817	5,414	—
Oxford . . . . M. & P.	4,933	5,230	27,843	27,561	—
Pembroke . . . . M. & P.	1,792	2,144	10,107	15,051	—
Penzance . . . . M.	1,878	1,968	9,214	9,414	—
Peterborough . . . . P.	1,755	2,509	8,672	11,732	—
Petersfield . . . . P.	1,072	1,135	5,550	5,655	—
Plymouth . . . . M. & P.	5,171	6,081	52,221	62,823	—
Pontefract . . . . M.	1,069	1,120	5,106	5,340	—
Pontefract P.	2,496	2,624	11,515	11,678	—
Poole . . . . M. & P.	1,903	2,036	9,255	9,745	—
Portsmouth . . . . M. & P.	12,825	15,803	72,096	94,546	—
Preston . . . . M. & P.	11,348	15,045	69,542	82,961	—
Reading . . . . M. & P.	4,098	4,851	21,456	24,965	—
Reigate . . . . M. & P.	792	1,581	4,927	9,975	—
Richmond . . . . M.	843	863	4,106	4,290	—
Richmond P.	1,032	1,046	4,969	5,134	—
Ripon . . . . M. & P.	1,345	1,382	6,080	6,172	—
Rochdale . . . . P.	5,829	7,701	29,195	38,164	—
Rochester . . . . M. & P.	2,549	3,171	14,938	16,672	—
Rye . . . . M.	726	784	4,071	—	3,794
Rye P.	1,557	1,682	8,541	—	8,202
Saffron Walden . . . . M.	1,173	1,246	5,911	—	5,474
St. Alban's . . . . M.	1,361	1,500	7,000	7,675	—
St. Ives . . . . M.	1,403	1,453	6,525	7,019	—
St. Ives P.	2,003	2,116	9,872	10,354	—
Salford . . . . M.	11,447	19,286	63,850	102,414*	—
Salford P.	15,342	19,286	85,108	102,414	—
Salisbury . . . . M. & P.	2,311	2,347	11,657	11,833	—
Sandwich . . . . M.	602	633	2,966	—	2,893
Sandwich and Deal . . . . P.	2,474	2,658	12,710	13,701	—
Scarborough . . . . M. & P.	2,838	3,987	12,915	18,380	—
Shaftesbury . . . . P.	1,894	1,928	9,404	—	8,987
Sheffield . . . . M. & P.	27,099	38,083	135,310	185,157	—
Shrewsbury . . . . M. & P.	3,900	4,360	19,681	22,055	—
Southampton . . . . M. & P.	5,749	7,718	35,305	46,970	—
South Molton . . . . M.	929	868	4,482	—	3,830
South Shields . . . . M. & P.	3,439	4,436	28,974	35,223	—
Southwark . . . . P.	23,751	25,683	172,863	193,443	—

* Since the Census of 1851, the Municipal Borough has been enlarged, and its boundaries are now the same as those of the Parliamentary Borough.

Table VIII.—*Houses and Population* (continued).

1		2	3	4	5	6
City or Borough.		Inhabited Houses.		Population.		
		1851.	1861.	1851.	1861.	
Stafford	M. & P.	1,977	2,241	11,829	12,487	—
Stamford	M. & P.	1,616	1,661	8,933	—	8,044
Stockport	M. & P.	10,568	11,286	53,835	54,681	—
Stoke-upon-Trent	P.	15,562	19,870	84,027	101,302	—
Stratford-upon-Avon	M.	694	787	3,372	3,672	—
Stroud	P.	8,182	8,196	36,535	—	35,513
Sudbury	M.	1,280	1,476	6,043	6,878	—
Sunderland	M.	7,975	10,222	63,897	80,324	—
	P.	8,519	11,216	67,394	85,748	—
Swansea	M.	6,001	7,931	31,461	42,581	—
Tamworth	M.	826	935	4,059	4,326	—
	P.	1,760	2,118	8,655	10,202	—
Taunton	P.	2,645	2,905	14,176	14,660	—
Tavistock	P.	1,009	1,134	8,086	8,804	—
Tewkesbury	M. & P.	1,274	1,265	5,878	—	5,876
Thetford	M. & P.	844	898	4,075	4,208	—
Thirsk	P.	1,154	1,208	5,319	5,351	—
Tiverton	M. & P.	2,181	2,209	11,144	—	10,444
Totnes	M. & P.	728	790	4,419	—	3,993
Tower Hamlets	P.	75,710	88,664	539,111	647,585	—
Truro	M. & P.	2,194	2,389	10,733	11,336	—
Tynemouth	M. & P.	4,295	4,883	29,170	33,991	—
Wakefield	M.	4,391	4,779	22,065	23,181	—
	P.	4,390	4,779	22,057	23,199	—
Wallingford	M.	522	553	2,819	—	2,786
	P.	1,635	1,662	8,064	—	7,794
Walsall	M. & P.	4,521	7,452	25,680	37,762	—
Wareham	P.	1,351	1,394	7,218	—	6,977
Warrington	M.	4,285	4,985	22,894	25,953	—
	P.	4,380	5,159	23,363	26,852	—
Warwick	M. & P.	2,229	2,273	10,973	—	10,589
Wells	M. & P.	906	860	4,736	—	4,648
Westbury	P.	1,535	1,519	7,029	—	6,495
Westminster	P.	24,755	26,430	241,611	253,985	—
Weymouth and Melcombe Regis	M. & P.	1,722	1,867	9,458	11,383	—
Whitby	P.	2,239	2,465	10,989	12,054	—
Whitehaven	P.	4,627	3,818	18,916	—	18,842
Wigan	M. & P.	5,686	6,698	31,941	37,657	—
Wilton	P.	1,721	1,799	8,607	8,674	—
Winchester	M. & P.	2,077	2,385	13,704	14,784	—
Windsor	M. & P.	1,417	1,570	9,596	9,827	—
Wisbeach	M.	2,141	2,086	10,594	—	9,275
Wolverhampton	M.	9,184	11,785	49,985	60,858	—
	P.	22,284	28,458	119,748	147,646	—
Woodstock	P.	1,623	1,664	7,983	—	7,820
Worcester	M. & P.	5,695	6,317	27,528	31,123	—
York	M.	7,077	8,243	36,303	40,377	—
	P.	7,778	9,162	40,359	45,326	—

TABLE IX.—*Islands in the British Seas. Houses and Population Enumerated in the Islands in the British Seas on April 8th*, 1861.

	Houses.			Population.		
	Inhabited.	Uninhabited.	Building.	Persons.	Males.	Females.
Islands in the British Seas	23,000	1,349	170	143,779	66,394	77,385
Isle of Man	8,948	502	90	52,339	24,544	27,795
Island of Jersey	8,327	381	46	56,078	25,304	30,774
Island of Guernsey and Adjacent Islands	5,725	466	34	35,362	16,546	18,816

TABLE X.—*Population in the Islands in the British Seas.*

	1821.	1831.	1841.	1851.	1861.
Islands in the British Seas	89,508	103,710	124,040	143,126	143,779
Isle of Man	40,081	41,000	47,975	52,387	52,339
Island of Jersey	28,600	36,582	47,544	57,020	56,078
" Guernsey (with Herm and Jethou)	20,339	24,540	26,698	29,806	29,846
" Alderney	No return	1,045	1,088	3,333	4,933
" Sark	488	543	785	580	583

Note.—Between 1831 and 1841 the population of the islands collectively increased about 18 per cent., and between 1841 and 1851 about 15 per cent. Between 1851 and 1861 a slight decrease took place in the Isle of Man and Jersey; while Guernsey would also have shown a decrease but for the circumstance of the military force in that island as well as in Alderney having been larger in 1861 than at the previous Census.

Table XI.—*Emigration (England and Wales). Return by the Government Emigration Board, showing the Number of Emigrants from the United Kingdom during the Ten Years* 1851 *to* 1861.

Year.	Total Number of Emigrants.	English.	Scotch.	Irish.	Foreigners.	Not distinguished.
1851......	335,966		No	return.		
'52......	368,764					
'53......	329,937	62,915	22,605	192,609	31,459	20,349
'54......	323,429	90,966	25,872	150,209	37,704	18,678
'55......	176,807	57,132	14,037	78,854	10,554	16,230
'56......	176,554	64,527	12,033	71,724	9,474	18,796
'57......	212,875	78,560	16,253	86,238	12,624	19,200
'58......	113,972	39,971	11,815	43,281	4,560	14,345
'59......	120,432	33,930	10,182	52,981	4,442	18,897
'60......	128,469	26,421	8,733	60,835	4,536	27,944
Total......	2,287,205	454,422	121,530	736,731	115,353	154,439

Note.—It appears from the above Table that 2,287,205 emigrants sailed from the ports of the *United Kingdom* in the ten years 1851-60. Of that number, 1,582,475 were emigrants in the eight years 1853-60; and 454,422 were found to be *English*, 121,530 *Scotch*, 736,731 *Irish*, and 115,353 *foreigners*. The origin of 154,439 emigrants was not ascertained. If we assume that these unascertained numbers should be distributed proportionally over the rest, the numbers of the several classes will be shown in the annexed Table.

The emigrants in the first three months of 1851 were 56,584, consequently the number in the two years 1851-2, namely 704,730, have to be reduced to 648,146, to obtain the number of emigrants in the interval extending from 1st April, 1851, to 31st December, 1852. The nationality was not then distinguished, but a large proportion of the emigrants was of Irish origin, and the best estimate will be made by assuming that the proportions were the same as in 1853; the results appear in the table. 18,734 emigrants sailed from the United Kingdom in the portion of 1861 terminating on April 7th, the Census day, and they have been distributed in classes in the proportions ascertained to exist in 1860.

Estimated Number of Four Classes of Emigrants in the Interval between the Census of 1851 *and* 1861.

Years.	Totals.	English.	Scotch.	Irish.	Foreigners.
Total........	2,249,355	640,210	183,627	1,230,986	194,532
1851 (April 1st)-1852	648,146	131,718	47,325	403,246	65,862
1853-60	1,582,475	503,568	134,674	816,408	127,825
1861 (to April 7th)	18,734	4,924	1,628	11,337	845

It may be inferred from the official returns that the emigration from the United Kingdom was not less than—

717,913 in the interval of the Censuses 1831 and 1841;
1,692,063 ditto ditto 1841 and 1851; and
2,249,355 ditto ditto 1851 and 1861.

On the Rule for finding the Value of an Annuity on Three Lives. By Augustus De Morgan, *Professor of Mathematics in University College.**

THE rule which was given by Thomas Simpson for finding the value of an annuity on three lives is as follows:—let A, B, and C be the three lives, and let K be a single life, the annuity on which is equal to an annuity on the joint lives of B and C; then the value of an annuity on the joint lives of A and K is the required annuity on the joint lives of A, B, and C. The life A should be the youngest of the three.

No reason has ever been given for this rule, simple as it is; and the only considerations of which I know, partaking of the nature of demonstration, and connected with it, are those by which Mr. Milne has shown that A should be the youngest of the three, and also that, instead of interpolating the age of K to a fraction of a year, the complete year of age next above the real age of K should be chosen.

If the rule be examined by the hypothesis of equal decrements, and if $1-a$, $1-2a$, &c., be the chances of A's living one, two, &c. years, and $1-\beta$, &c., $1-\gamma$, &c., the same for B and C, the error of the rule, or at least its order, will be represented by the fraction

$$\frac{2a\beta\gamma}{r^4}, \; (r = \text{interest of £1 for one year}),$$

whenever a, β, γ are so small that $a\beta\gamma$ is small compared with r^4. As far as my investigations go, I doubt if, on the hypothesis of equal decrements, the rule would be a good approximation when $a\beta\gamma$ is not a small fraction of r^4; and this may account for De Moivre not having noticed a rule which would, it might be imagined, present itself for trial at least to any one who was considering the subject for the first time.

A much more satisfactory account of this rule can be given from the expression for the law of human mortality which was given to the Royal Society, by Mr. Benjamin Gompertz, in 1825. That law is as follows:—the number living at x years from a given age (upwards of 10) may be represented by Ag^{q^x}, where Ag is the number living at the given age.

The constants do not retain their value during the whole of life—Mr. Gompertz, for instance, finds one set of values repre-

* This is the paper in the *Phil. Mag.* (Nov., 1839) quoted by Professor De Morgan at page 181, vol. viii., of this *Journal*, and referred to by Mr. Makeham in the Note at page 361, vol. ix.—Ed. *A. M.*

senting the Carlisle tables very nearly from 10 to 60, and another from 60 to 100. The closeness with which this theory represents the Northampton, Sweden, Carlisle, and Deparcieux's tables may be seen in the paper cited; and it adds not a little to the speculative value of the formula, that its author has deduced it from so simple a principle as that *the power of the human constitution to oppose decay loses equal proportions in equal times.*

If Mr. Gompertz's theory were accurately true, with a uniform value of the constants, throughout the whole remainder of life, Thomas Simpson's rule would no longer be an approximation, but an exact method. Let Ag^{q^x} be the number living at the age of A, and let B be y years older than A, and C z years older than B. Consequently the numbers now alive in the table at the ages of B and C will be

$$Ag^{q^{x+y}} \text{ and } Ag^{q^{x+y+z}}, \text{ or } Ah^{q^x} \text{ and } Ak^{q^x},$$

where $h=g^{q^y}$, $k=g^{q^{y+z}}$.

Hence the chances of the parties living t years are

$$g^{q^x(q^t-1)},\ h^{q^x(q^t-1)},\ k^{q^x(q^t-1)},$$

and, v being the present value of £1 to be received in one year, the value of the annuity is

$$(ghk)^{q^x(q-1)}\,.\,v+(ghk)^{q^x(q^2-1)}\,.\,v^2+\ \ .\ .\ .\ .\ .\ . \quad (1)$$

Determine w and l from $hk=g^{q^w}=l$, then it will be evident that, if (1) be called $\phi(ghk)$, we have $\phi(ghk)=\phi(gl)$, the annuity on the joint life of A and another, whose single life is worth $\phi(l)$ or $\phi(hk)$, the values of an annuity on the joint lives of B and C.

The various causes of error in the preceding, arising from the change in the values of the constants, will readily present themselves to those who have studied the subject.

It is not, of course, necessary that the progression of powers should be precisely that of Mr. Gompertz, and the following theorem may easily be demonstrated. If a_n be the chance of a life living n years, and if

$$a_n=(a_1)^{p_n},$$

where p_n is not a function of a, then the annuity on any number of lives is not altered in value if, instead of any part of those lives, a single life of equivalent value be substituted.

CORRESPONDENCE.

MR. EDMONDS: COLLEGE LIFE.

To the Editor of the Assurance Magazine.

My dear Sir,—In your last Number (p. 334) Mr. Edmonds has made some statements which, he says, it is "of importance that the reader should know." Unless he intend to insinuate that my exposure of his unfair treatment of Mr. Gompertz is the result of an old grudge, I cannot imagine how the matters he brings forward can concern any reader. But be the importance what it may, it is just as necessary that the reader should have it correctly as that he should have it at all. I therefore rectify mis-statements, destroy insinuations, and fill up suppressions.

1. That we were "not strangers" to one another at college—that we were of the same college, class-room, &c. *Not strangers* may mean anything: the fact is that we had not the slightest acquaintance, and that my full conviction is that we never exchanged a word during our joint college life (1823–1826). I cannot swear that no single word ever passed, but I have not the slightest recollection of it, nor belief in it.

2. That, in the third year, the (now) Astronomer-Royal was our lecturer; and that, towards the end of the course, the class was reduced to our two selves. The first statement is correct; my memory did not agree with the second: but as my old friend the Astronomer-Royal is well known for a preserver of documents, I wrote to him, and, begging he would excuse my saying why I wanted to know, I asked him what he could tell me about the attendance on his class of 1824–25, towards the end of the academical year. He replied that his *side* of the college had 20 students in the third year—that the attendance at the end was about $4\pm$; and he gave me the names of *eight* students (myself and Mr. Edmonds being two of them) from whom this average of four was recruited. Among these were three old friends of my own, whose attendance my memory tells me was tolerably regular.

3. That at a personal interview (in 1832) I expressed general approbation of Mr. Edmonds's work, but objected to one sentence, which I recommended him to cancel. That I did express such general approbation is perfectly true; but it is of importance the reader should know that I then had no knowledge of what Mr. Gompertz had done, except from Mr. Edmonds's "suppressive mention." The matter to which I objected, which should have been produced, since it is of at least as much importance to the reader as the fact that the Astronomer-Royal was our lecturer, is as follows:—Mr. Edmonds, finding that Mr. Finlaison's observations did not suit his theory, explained the discrepancy by saying (p. xiv.) that Mr. Finlaison was "a person whose qualifications for the task undertaken are unknown to the public." I thought this sentence, from a young gentleman then first appearing in the subject, savoured not a little of presumption; and, out of kindness to Mr. Edmonds, I endeavoured to induce him to cancel it: this he declined, as he says. He adds that on a subsequent occasion of communication with me he found that my favourable disposition towards him had ceased. I do not remember this second communication: if it took place after I had seen Mr. Gompertz's paper, I have no doubt that I made Mr. Edmonds feel the change of disposition which he describes.

4. That I have been, for 28 years, "secretly writing and speaking" against Mr. Edmonds. Private conversation is not "secret speaking" unless express means be used to make it so. I have never spoken about Mr. Edmonds in any but the usual way. As to the writing, the article "Mortality" in the *Penny Cyclopædia*, and also its authorship, were neither secret nor secrets.

One or two sentences have reference to the subject-matter of my accusation, about which I hold it needless to say any more. Should Mr. Edmonds favour you with any more college life, or other "secret" matter, I have no doubt you will insist on his making it appear what the reader's concern with it is before you insert it. One thing, however, I must particularly beg of you, namely, that if Mr. Edmonds should dare to make any definite assertion of the existence of any private grudge on my part, supported by any pretence of proof, no matter how trivial, you will print it all at once. Such publication will be due to me and due to the subject. Both of us have a right to any exposure of this kind which Mr. Edmonds may choose to make of himself.

Yours very truly,

August 6*th*, 1861. A. DE MORGAN.

P.S.—If you read Cooper's novels, you may remember that once, when somebody's Indian assailants were dispersed by the arrival of assistance, the party assailed did not think it worth while to send a shot after his enemies until he happened to see two of them in a line, when he could not resist the temptation of finishing them both with one barrel, merely from love of practice. I make a sort of imitation of this proceeding: there are two similar fallacies in one page, if not in one line, which both go down by one bullet. The form of logic under which they both come is as follows:—A is B, or rather it would have been if it had not been C. The first is, 'We attended the same mathematical class for three years—or, rather, we should have done so if I had not omitted to read mathematics during the whole of the second year." The second is contained in the last sentence of the page: speaking of the explanation of 1860, Mr. Edmonds proceeds, "which explanation, as Mr. De Morgan himself admits, would have exonerated me from the charge of 'unfair suppression' if it had been given in the year 1832." That is, I have admitted that the suppression of 1832 would not have existed if the statement made in 1860 had been made in 1832. Mr. Edmonds is correct—I did admit it; certainly if the proceeding of 1832 had been *description*, it would not have been *suppression*. But even here Mr. Edmonds suppresses something, and alters the grammar of the rest. Once more I have recourse to parallel columns.

The original (Jan. 1861, p. 214).	*The misquotation* (July, 1861, p. 333).
"If Mr. Edmonds had given all the description he has now given, weak as it is, there would have been foolish and unfounded self-assertion, but at least there would not have been suppression."	"If Mr. Edmonds had given all the description which he has now given . . . . there would have been no suppression."

MR. FINLAISON'S REPORT AND THE ENGLISH LIFE TABLE.

To the Editor of the Assurance Magazine.

Sir,—You will, no doubt, accord me a small space in your *Magazine* to enable me to offer a few words of explanation on the subject of a letter in your last Number from Mr. Bailey.

That gentleman, I consider, has quite lost sight of the scope and object of the letter which has become the subject of his criticism. I did not pretend to undertake an exhaustive analysis of Mr. Finlaison's elaborate Report: on the contrary, my letter was written at the period when the decennial census was on the very point of being taken, with the expressed object only of referring to such portions of the Report as had a bearing on the census question—with the view of showing in how many different ways the public might be affected by false or incorrect census returns.

I propose to offer a few very short remarks upon each head of Mr. Bailey's letter.

1. I am not answerable for Mr. Finlaison's "omission of eight words," by which the meaning of a passage in the Registrar-General's Report is stated to be altered.

Surely a person writing *currente calamo* upon a subject of popular interest is not bound to verify every statement contained in the subject-matter upon which he is forming his remarks, as if he were engaged upon an analytical investigation of a mathematical process.

2. As regards the over-estimate of age in very advanced life, I beg to observe that I do not admit, because a person is stated, in the census returns, or in those of the Registrar-General, to have attained the age of 100 years, that he has actually done so.

3. Mr. Bailey states that I have borrowed from my own imagination the description I gave of the manner in which certificates of death are frequently given by medical men. He says such gentlemen exercise no judgment whatever as to the ages of their patients.

I distinctly differ with him on this point, and beg to refer again to my account of the method adopted in practice to determine the age at death, the truth of which I still maintain.

4 and 5. I did not attempt to decide the question as to whether the vitality of females was or was not superior to that of males.

I confined my attention to showing what *might* be the effect, as regards the grant of life annuities and assurances, according as either sex might be shown to possess longevity superior to the other.

6. Upon the question of the discredit likely to be thrown upon the English Life Table in consequence of incorrect returns to the Census Enumerators or District Registrars, I can only reiterate my opinion that its value *must* be deteriorated in proportion to the number and magnitude of the errors included in the returns upon which it is based.

I expressed no opinion as to whether or not—any errors to the contrary notwithstanding—it was a better or a worse table than the Carlisle or any other table of mortality.

Mr. Bailey has, in fact, altogether misunderstood the intention of my letter, which was to point out to how great an extent the public are interested in the attainment of correct statistical information.

I thank my friend for the advice contained in his classically-concluding letter. I see that I must in future more frequently turn the stilus if I wish to avoid the criticism of so severe a Metius Tarpa.

I am, Sir,
Your obedient servant,
H. W. PORTER.

Alliance Assurance Office,
2nd Sept., 1861.

ON THE RECENT IMPUTATIONS MADE AS TO MR. GOMPERTZ'S ACCURACY.

To the Editor of the Assurance Magazine.

Sir,—The lengthy paper by Mr. Edmonds, which appeared in the last Number of the *Assurance Magazine*, renders it necessary that I should recur to a subject, which I should not again have approached if I had consulted my own inclination. In that paper, Mr. Edmonds has taken particular pains to answer some of my criticisms; and it will probably be correct to assume, that but for those criticisms this last paper of his would never have been written. Nevertheless my name is not once mentioned in that paper, but I am designated "the new advocate of Mr. Gompertz." If my remarks had been made anonymously, this would have been a perfectly natural and appropriate course for Mr. Edmonds to adopt; but under existing circumstances, I am unable to perceive what advantage the course adopted by him, has over the more obvious one of naming the person whose arguments he undertakes to meet. As there may, however, perhaps be some hidden merit in this course, to be discovered only upon trial, it may be worth while to try the effect of adopting some similar circumlocution. Thus I hope that in speaking of the "plagiarist of Mr. Gompertz," I shall be as well understood by your readers, as if I mentioned a name with which recent controversy will have rendered them familiar.

I have carefully read the paper by the plagiarist of Mr. Gompertz, and examined thoroughly all the arguments made use of by him in his defence; and I find that in one particular only, to be mentioned directly, is there any necessity to alter what I had previously written. Thus I might have rested content with making the single necessary correction, and referring the reader back to my unanswered arguments, without opening any fresh ground; and I should certainly have done so, had not the plagiarist of Mr. Gompertz taken the opportunity of throwing several fresh imputations upon the accuracy of Mr. Gompertz, equally unfounded with the former ones. In consequence of his having pursued this course, it becomes desirable to examine in detail both these new imputations and the answers to my former criticisms.

First, as to the point on which my former remarks require modification. I stated that I had come to the conclusion that the plagiarist of Mr. Gompertz had been "under a misconception as to the real nature of the process of integration." From his recent explanation (pp. 339, 340) I gather that he uses the phrase "process of integration" as equivalent to "solution" of the differential equation, and not, as I had supposed, in the more limited application to the step in which both sides of the equation are *integrated*—this step being (3) in my demonstration on p. 289. I therefore withdraw the particular remark as as to the "misconception" (p. 291); but in doing

this, I wish it to be particularly understood that I do not modify in the least the other remarks contained in the same paragraph. On the contrary, I find the conclusions I had arrived at singularly confirmed, and the observed phenomena at the same time explained, by a statement made by the plagiarist of Mr. Gompertz on p. 334, that he had omitted to read mathematics during the whole of the second year of his academic course. Those who are acquainted with the nature of the studies pursued at Cambridge, will at once admit how impossible it is for a student who has neglected to "read" during the second year of his course, to take up with any benefit or prospect of success, the subjects belonging to the third year. Thus I can now readily account for what seemed a very curious circumstance previously, that the plagiarist of Mr. Gompertz, having formed the differential equation $dy=-yap^x dx$, should have been unable to integrate it, as he frankly confesses (p. 178), although the process was "simple and well known."

Before proceeding to examine the new ground taken by the plagiarist of Mr. Gompertz, it will be convenient to review the answers he has put forward to my former remarks.

First, with regard to the constant d. I remarked that d is "an essential part of the formula, *and as such is employed by Mr. Edmonds himself.*" To this he replies (p. 340), "There is not the remotest approach to truth in this statement." It would be difficult to invent a more precise form of contradiction than is here employed. Let us see how far it is justified by facts. If the reader will turn to p. 336, he will find this identical letter d employed by Mr. Edmonds in the equation

$$\log. y=\log. d-\frac{ap^x}{\log. p}.$$

Again, if he will turn to p. 88, he will find the same quantity introduced, but there denoted by the letter g. How can the plagiarist of Mr. Gompertz say after this that he has not employed the quantity d? If d were really "superfluous" and "useless," it should be used neither in the process of solution of the differential equation, nor yet in the formula for L_x. It should therefore be omitted, and we should have simply $L_x=g^{q^x}$. But the plagiarist of Mr. Gompertz must admit this is not accurate, and that it would be impossible to deduce therefrom his favourite formula $10^{\frac{k^2a}{\lambda p}(1-p^x)}$. It makes no difference to the argument that the plagiarist of Mr. Gompertz has assigned a particular value to d in his formula. If he had not employed d at all, his formula would have been $10^{-\frac{k^2a}{\lambda p}.p^x}$, instead of that which he has actually employed. It should be noticed that this formula, which must be acknowledged by the plagiarist of Mr. Gompertz to be "defective," is very little more so, if at all, than his own, $10^{\frac{k^2a}{\lambda p}(1-p^x)}$. Neither gives the actual number of persons living at the age x. The former requires to be multiplied by the constant factor $L_0 . 10^{\frac{k^2a}{\lambda p}}$; and the latter by the constant factor L_0. On the contrary, Mr. Gompertz's formula, dg^{q^x}, does give the actual number living at the age x. It may be allowed to a person entertaining the views expressed by the plagiarist of Mr. Gompertz, to call

this formula "defective," as not being put in the form which appears to him the most complete and convenient; but there can be no possible pretext for terming it "erroneous," as is done on p. 336. I have thus shown that the formula is incomplete without d, and that if the plagiarist of Mr. Gompertz had not made use of d, his formula would have had a very different form. Therefore, I repeat, *d is not superfluous, and is employed by Mr. Edmonds himself;* and I leave it to those who are competent, to decide which of us is correct. Before leaving this point, it is worth while to notice the amusing inconsistency of the plagiarist of Mr. Gompertz, who (p. 181) says that the process of integration is rendered obscure and ambiguous by the aid of the superfluous and useless indeterminate constant d, and yet himself introduces this very constant as I have shown.

Nor, secondly, is d "indeterminate." It would appear (*v.* p. 338) that the plagiarist of Mr. Gompertz has purposely employed this word in a sense different to that to which it is usually limited. To this course there is only one objection, viz., that a writer pursuing it is certain to be misunderstood. This may be obviated by a process which tradition affirms to have been employed in the "schools" at Cambridge. A student under examination made the error of writing 4 instead of 3 throughout a mathematical demonstration. Arrived at the conclusion, he discovered his error; but being pressed for time, instead of correcting the demonstration throughout, he appended a note—"N.B. Throughout the above demonstration the symbol (4) denotes three units." So, the plagiarist of Mr. Gompertz should say, "By the word *indeterminate*, I denote, not, as is commonly understood, that of which the value cannot be determined, but"—something quite different. Such a course may be thought by some persons to have its recommendations; and the meaning of the writer might be ascertained by the use of sufficient care. But the use of a word without comment in a sense different from that attached to it by all other persons, is wholly unjustifiable!

The plagiarist of Mr. Gompertz argues (p. 337) that d is rightly called *indeterminate* in the usual sense, because (he asserts) "Mr. Gompertz and his two advocates" have been unable to determine its value. If this were true, still he forgets that since *he* has succeeded in getting the value of d, it can no longer be called indeterminate. But as to the question of fact, he would have found, if he had read two lines further (p. 291), that Mr. Gompertz's "new advocate" had succeeded in determining the value of *d!* He would have found the equation $L_n = dg^{q^n}$, whence $d = \frac{L_n}{g^{q^n}}$, which reduces to the form given by the plagiarist of Mr. Gompertz $\left(\frac{L_0}{g}\right)$ on making $n=0$. This is a misstatement of fact, of which, whether intentional or accidental, I have just reason to complain. I regret to say that it is not the only instance of the kind which appears in the paper now under consideration.

I next say that the formula $10^{\frac{k^2 a}{\lambda p}(1-p^x)}$ is defective, because it does not contain the factor L_0; and I point out, by a reference to a table of mortality calculated and published by the plagiarist of Mr. Gompertz, that he in practice supplies the defect of his formula. It will be seen by any person who is at the trouble to refer to the table A.1. of Mr. Edmonds's *Life Tables*, that $L_{12} = 100{,}000$ and $L_{22} = 92865{\cdot}8$; whereas if "the

superfluous quantity L_0" (here L_{12}) is omitted from the formula, we should have $L_{22}=\cdot 928658$. How far this can be properly described as reproducing "the identical figures given as obtained by the substitution of unity for L_0" (*v.* p. 341), I again leave the reader to decide.

The next point to which I drew attention was that the plagiarist of Mr. Gompertz had committed the serious error of calling d a particular value of y. He confesses to this error (p. 338), saying that "in strictness" he should have said something quite different to what he did actually say. Having made an undeniable mistake, it is only proper to acknowledge it when pointed out. Why then does he, nine lines further down, retract that confession by speaking of "the *pretended* verbal error"? This appears to me at least a great want of candour; for the error is beyond doubt a real, not a pretended one, whether it is only "verbal" or something more. The plagiarist of Mr. Gompertz expresses surprise that while pointing out his errors, I have not noticed a "really serious error of Mr. De Morgan." The reason is easily given. I have no ambition to become a public censor. I have, as stated in my former letter, a definite object in view in exposing the errors of the plagiarist of Mr. Gompertz. I expect to demonstrate by doing this, that his imputations on Mr. Gompertz, besides being unfounded, are without weight, as coming from a person whose acquaintance with the subject is imperfect. If Professor De Morgan had committed a really serious error, no similar reason existed why I should point it out, and no good object was to be gained thereby.

Further on, I show that the plagiarist of Mr. Gompertz has himself changed the sign of the quantity c, which he had asserted that Mr. Gompertz changed. The truth of my statement is not denied; but the plagiarist of Mr. Gompertz appears to have had this remark in view while writing the second paragraph on p. 337. He there says that Mr. Gompertz has changed the sign of the exponent of g. This is a phrase which is literally unmeaning. We have $g=10^c=10^{-\frac{k^2\alpha}{\lambda q}}$, where it is obvious that the "exponent of g" is unity;* and how its sign is changed by Mr. Gompertz, or can be changed at all, it is impossible to imagine. What the plagiarist of Mr. Gompertz apparently meant to state, was, that Mr. Gompertz committed an error in making the above supposition as to the values of g and c, instead of taking $g=10^c=10^{+\frac{k^2\alpha}{\lambda q}}$. When stated in this plain way, and cleared of the mist of verbiage in which it has been diligently involved by the plagiarist of Mr. Gompertz, the whole objection is obviously futile. Mr. Gompertz was clearly at liberty, without committing an error or introducing a "defect," to make whichever supposition he pleased as to the values of g and c. The one supposition would have given him the formula, $\frac{d}{g^{q^x}}$, instead of that which he has actually employed, dg^{q^x}. The plagiarist of Mr. Gompertz considers the former of these formulæ to be preferable. Mr. Gompertz preferred the latter: and so, I believe, will all your readers who consider the point.

Here I would call attention to a curious inconsistency between the first and third paragraphs on p. 337. The first states that the equation

* It seems probable that the quantity c is intended by this term, but c is the *common logarithm* of g, not its exponent.

$y=dg^{p^x}$ is defective because it has not introduced the quantity L_0 by putting $\frac{L_0}{g}$ for d; the third asserts that L_0 ought *not* to have been introduced, and that the formula is defective because use *has* been made of L_0! It is, of course, obviously impossible that both these charges can be true, and the slightest examination of Mr. Gompertz's process will satisfy the reader that it is not chargeable with the "defect" of introducing L_0. I cannot leave this point without noticing that L_0 is incorrectly introduced into Mr. Gompertz's process by his plagiarist; for, as I have already pointed out (p. 291), Mr. Gompertz takes x throughout to denote the age *measured from birth*, so that the correct quantity to introduce is L_n, and not L_0.

The last of my remarks to which the plagiarist of Mr. Gompertz has attempted a reply, has reference to a misquotation by him of Mr. Gompertz's words. I will not occupy your space by again quoting the passage misquoted, but refer those who wish to verify what I state, to p. 293. The plagiarist of Mr. Gompertz has quoted a portion of a hypothetical sentence, which was not intended as a formal statement of the law of mortality, and argues on the strength of the misquoted passage, that Mr. Gompertz has committed an error, in not stating that his law of mortality does not prevail throughout the whole of life with the same constants. The truth being, that when Mr. Gompertz does state his law of mortality, he is very careful to limit it to a "long portion of life." I could have easily supposed that the misquotation was made through haste, but it is actually defended in the recent paper of the plagiarist of Mr. Gompertz! It is described as a simple omission of the "insignificant word" *if!* This is the first time I have ever heard the word *if* described as insignificant. The point will be made clearer by an example. Thus I may say, "*If* Mr. Edmonds has knowingly misrepresented facts in order to deprive Mr. Gompertz of the credit to which he is justly entitled, *then* he will not be correctly described by the term 'plagiarist,' but a far stronger term must be applied to him!" Suppose this passage to be quoted thus:—" Mr. Edmonds has knowingly misrepresented facts in order to deprive Mr. Gompertz of the credit to which he is justly entitled;" and we shall have an exact parallel to the misquotation with which I charge Mr. Edmonds. Perhaps this example may convince that gentleman that the little word "if" is not so "insignificant" as he has hitherto imagined. But it seems (p. 340) that "if" is used by Mr. Gompertz in the sense "let it be assumed." This I deny; but supposing it were, I cannot see that it would in the least affect the question.

The simple facts of the misquotation are these. Mr. Gompertz carefully limits the application of his hypothetical law of mortality to a "long portion of life." His plagiarist omits all reference to the passage in which this limitation is made—whether from design or oversight I will not pretend to say—but quotes part of a sentence in an altered form, as a proof that Mr. Gompertz has erroneously omitted to limit the application of his law. It is not true, as the plagiarist implies, that the only difference between the passage as written by Mr. Gompertz and quoted by him, consists in the presence or absence of the word "if". The reader will see, by comparing the passages (pp. 174, 293), that "if" is replaced by "that"; "were" is replaced by "are", and "lost" by "loses"; while in further illustration of the looseness with which the plagiarist of Mr. Gompertz has made the quotation, it may

be mentioned that he wholly omits the significant word "remaining", as applied to the "power of resisting destruction" by Mr. Gompertz.

A similar looseness exists throughout the recent papers of the plagiarist of Mr. Gompertz. In one instance, I have to complain that my words are misquoted. I am represented (p. 339) to have said that "the process of integration had not *commenced* when the erroneous (b) first appeared." On referring to p. 291, it will be seen that my actual words are, "b makes its appearance in step (2), while the integration is not *performed* till (3)." By the substitution, whether from design or carelessness, of the word *commenced* for the original one *performed*, a totally new complexion is given to the sentence, and the original meaning is obscured and perverted.

Again, it is stated (p. 338) that "Mr. Gompertz has *recently* presented to the Institute of Actuaries a corrected copy of his paper." The indefinite word *recently* is calculated to leave a false impression on the mind of the reader—whether designedly or not, I will not pretend to say. It would naturally be supposed, on reading the passage from which I have quoted, that the corrected copy of Mr. Gompertz's paper had been presented to the Institute of Actuaries, since attention has been drawn to that paper by the communication of Professor De Morgan, which appeared in the *Assurance Magazine* for July, 1860. So far from this being the case, I have ascertained that the copy was presented so long ago as February, 1855. It is true that this may be termed *recently* in comparison with the date (1825) of the original paper; but this very circumstance renders the use of the ambiguous term more objectionable.

As another example of the same kind, I would refer to the description given by the plagiarist of Mr. Gompertz, of the problem proposed for solution. He says (p. 335; see also p. 338) that the object of Mr. Gompertz was "to find an expression for the number living at any age (x) in terms of the annual mortality." This is incorrect. *This was Mr. Edmonds's object, but was not Mr. Gompertz's.* The object of the latter may be briefly expressed as follows:—The mortality (not the *annual* mortality) at any age x being proportional to q^x (*i.e.* $=aq^x$), required a formula for the number living at the age x. Mr. Gompertz's formula (dg^{q^x}) gives the correct solution of his problem, and Mr. Edmonds's formula, *corrected by the introduction of* L_0 (thus, $L_0 . 10^{\frac{k^2 a}{\lambda p}(1-p^x)}$), gives the correct solution of *his* problem.

The plagiarist of Mr. Gompertz expresses his belief (p. 333) that that gentleman has never complained that any wrong has been done him by a failure of due acknowledgment of the priority of his discovery. Here I would direct attention to the gradual change of tone in the paragraph relating to this point. First it is "I do not believe," then "I have never heard," and finally "Mr. Gompertz himself, during the period of 28 years, remained insensible to the supposed wrong." Truly a very positive statement to base upon so slender a foundation of fact, or rather belief! With reference to this point, I happen to have heard from a gentleman, himself a Fellow of the Royal Society, that Mr. Gompertz has, upon one occasion at least, in the course of conversation at the Royal Society, some twenty-five years ago, expressed in emphatic language his belief that his theory had been adopted without proper acknowledgment. It is easy enough to understand that gentlemen who have been "in free personal communication" with both

Mr. Gompertz and his plagiarist, and therefore, it is to be presumed, on friendly terms with both, would not make known a circumstance of this kind to the latter; and this consideration may suggest to that gentleman that it is not safe to assume that an event has not taken place because he has not heard of it.

The plagiarist of Mr. Gompertz states (p. 334) that his paper of Oct., 1860, "was written in self-defence (against an outrageous attack of Mr. De Morgan)." It appears to me that that paper contained an outrageous and unprovoked attack upon Mr. Gompertz, and that the more recent paper of last July contains another "outrageous attack" upon the same gentleman.

Thus a very laboured effort is made (p. 335) to show that Mr. Gompertz has fallen into a serious error with regard to the nature of the quantity a in his formula aq^x for the mortality at any age x. There is not the slightest foundation in fact for this charge, as I will now proceed to prove. It is by no means easy to perceive at first sight the precise meaning to be attached to the quantities a and α as employed by the plagiarist of Mr. Gompertz, in consequence of the obscurity of the language he has employed. I believe, however, that the following will be found to be a clear and faithful exposition of the distinction he means to draw. Adopting for the present his own letters, we have αp^x for the rate of mortality at the age x. The meaning of this quantity is best seen by the application that is made of it: thus the number dying in the small time dx, out of L_x alive, is stated to be $L_x \alpha p^x dx$. Whence we see that, in mathematical language, $L_x \alpha p^x$ is the limit of the ratio which the number dying in a small time bears to that time, when both are indefinitely diminished. If we represent the number alive at age x by unity, this ratio becomes αp^x, and at the age 0 it is simply α. Now in making a comparison with any actual table of mortality, we naturally take the *annual* mortality at the age considered. This will not be represented by α, *i.e.* if the rate of mortality as explained above is α at the commencement of the year, and is supposed to remain constant throughout the year, the number who die in the year out of L_0 persons living at the commencement will not be αL_0. In order to determine the annual mortality we must proceed as follows:—the number alive at the commencement of the year being L_0, suppose the year divided into ν equal portions, then the number dying in the first small interval $\frac{1}{\nu}$ will be $\frac{L_0\alpha}{\nu}$. Thus there remain alive $L_0\left(1-\frac{\alpha}{\nu}\right)$ persons, and of these $L_0\frac{\alpha}{\nu}\left(1-\frac{\alpha}{\nu}\right)$ will die in the next interval $\frac{1}{\nu}$, leaving $L_0\left(1-\frac{2\alpha}{\nu}+\frac{\alpha^2}{\nu^2}\right)$ $=L_0\left(1-\frac{\alpha}{\nu}\right)^2$ alive at the end of the second interval. Pursuing the same process, it is easily seen that the number surviving at the end of the year must be $L_0\left(1-\frac{\alpha}{\nu}\right)^\nu$. Now pass to the limit by supposing ν indefinitely large, and this quantity becomes $L_0e^{-\alpha}$. Thus the number dying in the year will be $L_0(1-e^{-\alpha})$, whence the quantity $1-e^{-\alpha}$ represents the *annual mortality*. The plagiarist of Mr. Gompertz supposes the annual mortality to be such that out of $1+a$ persons alive at the beginning of the year, a die in the course of the year, so that the annual mortality is $\frac{a}{1+a}$. Thus

then he gets $\dfrac{a}{1+a}=1-e^{-\alpha}$, whence $e^{-\alpha}=1-\dfrac{a}{1+a}=\dfrac{1}{1+a}$, and $1+a=e^{\alpha}$; also $\alpha=\log_e(1+a)$.

The only argument put forward in order to support the charge against Mr. Gompertz that he was ignorant of the real nature of the quantity α, is that he has made no statement of "any algebraical or arithmetical value of a or α." It is difficult to attach any precise meaning whatever to these words; but the thing intended appears to be this:—Mr. Gompertz has not explained the precise meaning of his quantity a, nor given any numerical example from which that meaning might be inferred—*therefore* he was ignorant of its meaning! The truth is that for Mr. Gompertz's object of comparing the survivors at any age with his formula dg^{q^x}, it was not necessary to make any inquiry as to the nature of the quantity a. But this affords not the slightest ground for supposing he was ignorant of its nature.

It is true that there is no unqualified assertion made that Mr. Gompertz is in error on this point. We read however (p. 336), that "the differential equation of Mr. Gompertz is erroneous *if* he has used the finite decrement (a) to represent the mortality;" see also p. 340. Now in reading this sentence, it must be borne in mind that the plagiarist of Mr. Gompertz looks on the little word *if* as an insignificant word, so that his meaning must be that there is very little doubt indeed, in fact none at all, that Mr. Gompertz was in error. We have seen that there is no foundation whatever for this assumption, and that Mr. Gompertz in this, as in all the other points I have considered, is strictly accurate. Let us now turn and consider more closely the remarks of his plagiarist on this point, and there will remain no doubt that he has fallen into two very distinct and serious errors.

In the first place, he has taken to denote the annual mortality a quantity a such that the number living at the commencement of the year is to the number living at its end as $1+a$ is to 1. Instead of this, the correct assumption is to take a so that the number entering on the year is to the number surviving it as 1 to $1-a$. This is a fundamental error, and one that was not to be expected from the veriest tyro in actuarial science. To put the point in another way:—the annual mortality at any age x is found by comparing the finite decrement for one year with the number living at the *beginning* of the year; whereas the plagiarist of Mr. Gompertz has compared it with the number living at the *end* of the year! The practical result of this error will be, that supposing α to be the mortality at the beginning of a year, the plagiarist of Mr. Gompertz makes the actual mortality of the year greater than α, instead of less as it obviously should be. To take the figures given by him in p. 335, if $\alpha=\cdot0063643$, then will the actual mortality of the year be $1-e^{-\alpha}$ or $\cdot0063440$, instead of $\cdot0063845$ as stated.

Again, in calculating the actual mortality of the year, the plagiarist of Mr. Gompertz supposes the rate of mortality constant throughout the year. This is contrary to the fundamental hypothesis, that the mortality at any age x, integral or fractional, is ap^x. The mortality varies throughout the year, and the actual annual mortality will be neither $e^{\alpha}-1$ as the plagiarist of Mr. Gompertz has it, nor $1-e^{-\alpha}$ as found by me above. The real annual mortality is found by a very simple process, for it is equal to

$$\frac{L_x-L_{x+1}}{L_x}=1-\frac{L_{x+1}}{L_x}.$$

But $$L_x = dg^{p^x},\ L_{x+1} = dg^{p^{x+1}};$$

whence $$\frac{L_{x+1}}{L_x} = \frac{g^{p^{x+1}}}{g^{p^x}} = g^{p^{x+1}-p^x} = g^{(p-1)p^x}.$$

Also $g = 10^e = 10^{-\frac{ak^2}{\lambda p}}$; so that the correct value of the annual mortality in terms of known quantities is

$$1 - g^{(p-1)p^x} = 1 - 10^{-\frac{ak^2(p-1)}{\lambda p}\cdot p^x};$$

which becomes when $x=0$, $1 - 10^{-\frac{ak^2(p-1)}{\lambda p}}$.

Before leaving this point, I would call the attention of your readers to the circumstance that the plagiarist of Mr. Gompertz has termed a (p. 335) the most important quantity in the formula dg^{q^x}. What! Is it more important than q, which is "part of the foundations of the universe"? (*V.* p. 177.) We have here a remarkable instance of a complete revolution in opinion in the course of nine months. In October, 1860, it was stated that q is "distinguished from other but *less important* constants used"; in July, 1861, it appears that a and not q is the most important constant!

On the same page we read:—"Previous to forming the differential equation, it is essential that the nature of the quantity (a), given to represent the proportional mortality for one year when the mortality is constant, should be determined." With regard to this, I simply remark that the success with which Mr. Gompertz has applied his formula to the examination of various tables of mortality, without previously examining into the nature of a, sufficiently shows that it is *not* essential that the nature of a should be determined. Secondly, I would particularly draw attention to the fact that a is *not* "given" by Mr. Gompertz "to represent the proportional mortality for one year′ when the mortality is constant." He nowhere throughout his paper speaks of the mortality being *constant for one year;* but on the contrary consistently supposes the mortality to vary at every instant of life; and he nowhere speaks of a being the mortality *for one year.* His words are that "the intensity of his mortality at the age x may be denoted by aq^x." In this instance, as in several others, Mr. Gompertz's words are wholly misrepresented. Before leaving this point, I may remark that it is quite impossible the annual mortality should be expressed by two different quantities (*v.* p. 335). If the one quantity correctly expresses the annual mortality, it is obvious the other cannot do so too. I have already shown that *neither* of the two quantities described by the plagiarist of Mr. Gompertz correctly expresses the annual mortality.

Not only is Mr. Gompertz's accuracy called in question, but his truthfulness is also impeached. We are told (p. 330) that Mr. Gompertz has lent "his approbation to papers containing *statements which, from his superior knowledge of the facts, he could not, with any regard to truth, have uttered himself.*" Unfortunately, we are left to conjecture what those untrue statements are. For myself, I can find no statement, either in Professor De Morgan's paper or my own, to which that description will apply. It is therefore impossible to disprove this serious charge; but I feel not the slightest doubt, that all your readers will agree with me in considering that the plagiarist of Mr. Gompertz has pursued an unwarrantable course, in recklessly making this grave charge without the least

attempt to substantiate it. Similarly, we are told that "his advocates advance claims on his behalf, which Mr. Gompertz has never advanced himself." Here again, no attempt is made to point out these claims, and I am therefore only able to give a general denial to the assertion. I can discover no new claims put forward on behalf of Mr. Gompertz by Professor De Morgan and myself, and I do not believe that any such have been put forward. So, again, we are told there is only one instance of "concurrence of views" between Mr. Gompertz and his two advocates, "in the recently published papers of the three parties." It would have been more to the point if the plagiarist of Mr. Gompertz had given any example of discordance of views between "the three parties." Probably he has not done so, because he was unable to find any. With regard to the particular point which he has noticed as one of agreement, viz., the statement that he has given no proof of the truth of his supposed law of mortality, I have several times carefully read Mr. Gompertz's letter which is contained in the *Assurance Magazine* for April, 1861, and have failed to discover that he makes any statement of the kind at all. In fact, after carefully reviewing the whole of the remarks on p. 330, the conclusion appears unavoidable that they were written with far greater regard to effect than to truth.

A great deal has been said about the errors which appear in Mr. Gompertz's process, as printed in the *Philosophical Transactions.* I had hoped that I had satisfactorily shown that those errors are simply errors of the press, and that the process itself is free from any intrinsic error. In this view I am supported by the manuscript corrections in the copy of Mr. Gompertz's paper belonging to the Institute of Actuaries, and by Mr. Gompertz's own remarks in the *Assurance Magazine* (p. 297). Even without such support, I submit that when a gentleman of undoubted mathematical ability starts with a correct equation $a\mathrm{L}_x q^x \dot{x} = -\dot{(\mathrm{L}_x)}$, and ends with a correct solution $\mathrm{L}_x = dg^{q^x}$, it is only reasonable to suppose that he is aware of the correct intermediate steps, and that any errors which appear in the published process arise—not from ignorance or carelessness on the part of the author—but from a want of proper supervision of the press. On one point I venture to differ with Mr. Gompertz. He wishes for "a sufficiently clever mathematical superintendent of the press." On the contrary, I am of opinion that it is preferable to have a superintendent of the press who is ignorant of mathematics, but familiar with mathematical symbols. For it is clearly impossible to obtain one with sufficient mathematical knowledge to "restore" a demonstration such as Mr. Gompertz's published one; and a person of imperfect knowledge, who attempts to make conjectural emendations, is likely to introduce more new errors than he corrects. On such grounds I have been given to understand that women are often found better compositors of Latin and Greek works than men, because being quite ignorant of the language, they simply set up what is put in their hands, and do not introduce errors by attempting to correct fancied mistakes. Thus the whole responsibility of correcting the press is left to the author. It appears from the remarks on pp. 338–9, that the plagiarist of Mr. Gompertz does not yet allow that the second b is a printer's error. He has put forward a conjecture (p 339) to account for the introduction of b, which supposes a degree of ignorance or carelessness on the part of Mr. Gompertz, which is quite inconceivable on the part of a gentleman of such proved mathematical attainments. Mr. Gompertz is represented as having erro-

neously supposed that because a constant was introduced by integration on one side of the equation, therefore another was required on the opposite side! This is wholly opposed to the fundamental notion of integration, by which *one* constant only is introduced; and it is clearly impossible to entertain the supposition for an instant. This theory has been put forward in apparent forgetfulness of a different explanation that was volunteered (pp. 181, 184) to account for the appearance of b. Both of these explanations cannot be correct, and I believe neither is.

I would ask the plagiarist of Mr. Gompertz how he accounts for the corrections made by Mr. Gompertz's authority in the copy of his paper belonging to the Institute of Actuaries—made, it should be remembered, before February, 1855? This at least shows that Mr. Gompertz at that date repudiated the letter b as a misprint, and as forming no part of the correct process. When does his plagiarist suppose he became aware of the error? This error is exactly such a one as I should expect to see introduced by a compositor who knew something of algebra. The letter b having been introduced in the previous step erroneously, and the equation $a\dot{x}q^x = -\frac{\dot{L}_x}{L_x}$ having been written $abq^x = -\frac{\dot{L}_x}{L_x}$, such a compositor would naturally argue—"the letter b must have been forgotten by the author in "the next step. Instead of aq^x, which the MS. shows in the next step, we "should clearly have abq^x, as in the previous one." It must be borne in mind that the plagiarist of Mr. Gompertz allows the introduction of b in the former of these cases to be a genuine misprint.

In the same passage (p. 339) occurs a statement for which I am wholly unable to account:—"The equation, as corrected, is more defective, in form, at least, than was the equation intended to be corrected. . . . . The correction now made would leave no new constant on either side of the equation." The corrected equation in question will be found on p. 289 indicated by (3), *and contains the new constant* (d). I shall feel obliged if the reader will refer to this passage and verify my statement, and then endeavour to reconcile it with the one I have quoted above.

On the same page it is stated that I do "not call in question the truth of the substantial part of the statement" that "b is introduced through an incorrect process of integration, and is superfluous, useless, and erroneous." I certainly intended to call in question the truth of this. I have stated and, I believe, proved, that Mr. Gompertz's process is not erroneous, and that b is not a superfluous, useless, and erroneous constant, but is simply a misprint. It appears to me that I have thus, in a sufficiently plain manner, called in question the truth of the statement, not only of its form but of its substance.

Besides the imputations on Mr. Gompertz's accuracy, we have others on his language and style (p. 329) to which it is only desirable to refer in order to show the spirit which animates the remarks now under review. I should be glad to learn how it can affect the questions at issue, whether Mr. Gompertz's language is "homely" or "transcendental." Surely it must be a very weak case which needs the support of such an argument as is here implied!

In my former letter I showed the necessity of detailed evidence before admitting the truth of the supposed "true law of mortality." In consequence of this challenge apparently, we have now set forth (p. 331) a short

table comparing the mortality among the male population of England during 10 years according to observation and according to theory. On this I would remark that the agreement of the two does not proceed in any part of the table beyond the first decimal place, so that we are still left in ignorance as to the materials from which the values of p were computed to *seven* decimal places, and the logarithms to *four* decimal places. In addition to this, a large number of instances must be produced before it could be allowed that the law is commonly, much less universally, true. I would also ask whether the plagiarist of Mr. Gompertz will assert that the example he has produced is not equally in favour of the new law of mortality as recently stated by Mr. Gompertz?

Besides the errors which I have already shown are chargeable to the plagiarist of Mr. Gompertz, his language in various passages renders it extremely probable that he has fallen into another serious error. He says (p. 328) that his law of mortality is such that if the annual mortality at age 25 were 1 per cent, and at 45 were 1·80 per cent, then the annual mortality at the intermediate ages would be found by interpolating the terms of a geometric progression. Again, on p. 329, he says that "the annual rate of mortality increases in a geometric progression, of which the common ratio is (p)." Now, bearing in mind the distinction already pointed out, viz., that if ap^x be the rate of mortality at the age x, then the *annual mortality* at that age will be $1-g^{(p-1)p^x}$, it is obvious that the annual rate of mortality does not increase in geometric progression; but on the contrary, the chance of living a year $(=g^{(p-1)p^x})$ decreases in a geometric progression.

After all that has been already said, it is needless to comment at any length on the remarks contained in p. 330 and elsewhere, to the effect that Mr. Gompertz's method of procedure, in comparing the numbers living at various ages with those given by his formula dg^{q^x}, is not so good as the method of comparing the mortality in successive decennial periods of age. This simply amounts to saying that Mr. Gompertz's method is inferior to that of his plagiarist. This is a point upon which difference of opinion may fairly exist. The plagiarist of Mr. Gompertz should therefore have given it as his opinion, not asserted it as a fact, that the one method is preferable to the other. We must not omit to notice however the curious statement on p. 329, that the formula $10^{\frac{k^2a}{\lambda p}(1-p^x)}$ is "dg^{p^x} corrected and reduced to its *simplest* terms." It might be thought that it would be allowed on all hands that the latter formula is far simpler than the former; but it seems that long continued admiration of the former has led its author to claim for it merits which it certainly does not possess.

In conclusion, I beg to state that I regret extremely the personal character which this controversy has taken, but it has been forced upon me. When general and unsupported charges of inaccuracy are preferred, such as those which I have already pointed out, and such as that preferred against myself on p. 340, it becomes necessary to examine into the accuracy and knowledge of the person who makes them. If he is found to be a person free from error and accurate in his writings, his charges, although unsupported, will have weight with the world; but if on the other hand he should be found singularly inaccurate, then his charges will be disregarded and fall harmless to the ground, or recoil upon himself. Bearing these facts in mind, I have endeavoured to show the credit which I think should

be attached to the charges preferred by Mr. Edmonds against Mr. Gompertz, and I must leave it now to the scientific public to pronounce their verdict.

I have the honour to be, Sir,
Your obedient servant,
T. B. SPRAGUE.

25, *Pall Mall*,
September, 1861.

ON THE TABLES OF DEFERRED ANNUITIES PUBLISHED BY THE NATIONAL DEBT OFFICE.

To the Editor of the Assurance Magazine.

Dear Sir,—Having had occasion to refer to the above-mentioned Tables, I have been surprised to find that they are not computed in the usual way, but apparently on some principle wholly different from it. I append instances in respect of single premiums (but the same remark applies to the annual ones), in which your readers will observe that the premiums charged by the Government are, for the most part, greatly in excess of those resulting from calculations made on the true principles. The publication of these discrepancies may possibly bring about some explanation of the reason of them; and I therefore have to beg the favour of your inserting this communication in your *Journal;* remaining—

Dear Sir,
Yours truly,
J. W. STEPHENSON.

London, August, 1861.

Deferred Annuities of Thirty Pounds—Males.

Single Premiums, returnable without Interest at any time prior to commencement of Annuity, pursuant to 16 & 17 Vict., cap. 45.

Age at Entry.	Term.	Government Premiums.			Premiums computed on Government Data at 3 per Cent.			Difference.			Difference per Cent.
		£	s.	d.	£	s.	d.	£	s.	d.	
21	After 10 years	403	12	5	403	5	0	0	7	5	
,,	,, 20 years	261	15	0	252	11	10	9	3	2	3·63 per cent.
,,	,, 30 years	156	17	6	140	18	10	15	18	8	11·3 ,,
,,	,, 40 years	89	12	6	66	14	5	22	18	1	34·33 ,,
,,	,, 50 years	44	7	6	23	9	7	20	17	11	89 ,,
31	After 10 years	360	5	0	360	14	6				
,,	,, 20 years	216	0	0	207	13	3	8	6	9	4·01 per cent.
,,	,, 30 years	121	7	6	101	17	6	19	10	0	19·14 ,,
,,	,, 40 years	63	12	6	37	6	0	26	6	6	70·57 ,,
,,	,, 50 years	24	5	0	7	7	6	16	17	6	228·8 ,,
41	After 10 years	297	7	6	297	5	11	0	1	7	
,,	,, 20 years	167	2	6	153	5	10	13	16	8	9·02 per cent.
,,	,, 30 years	84	2	6	59	8	3	24	14	3	41·59 ,,
,,	,, 40 years	34	15	0	12	8	5	22	6	7	179·79 ,,
51	After 10 years	230	0	0	225	14	8	4	5	4	1·88 per cent.
,,	,, 20 years	115	17	6	95	7	6	20	10	0	21·49 ,,
,,	,, 30 years	47	17	6	21	19	1	25	18	5	118·33 ,,
56	After 25 years	56	2	6	30	4	10	25	17	8	85·69 per cent.

INSTITUTE OF ACTUARIES.

SOLUTIONS OF THE SECOND YEAR'S EXAMINATION QUESTIONS.

Some years ago, Mr. Porter, with a view to assist candidates for the Institute's certificates, gave solutions of the questions proposed to them, which, it may be remembered, were published in the first and second volumes of this *Journal*. Since these solutions appeared, modifications have, on more than one occasion, been made in the syllabus put forth by the examiners, and corresponding changes have consequently arisen in the order and character of the questions proposed. Under these circumstances, we willingly adopt the suggestion made by Mr. Sprague, who was one of the examiners of the last year, as to the desirableness of again publishing some solutions of the questions submitted of late to the candidates, so that future ones may have the best information as to the kind of answers expected from them; and having been favoured by Mr. Sprague with the following solutions of some of the questions proposed in the second year's examination of 1860, and with the references he has made, as regards others, to such text-books as are likely to be most useful to the student in his preparation, we now lay them before the reader. The questions themselves appeared in the last volume, see page 301.

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1. "Assuming the formula for $\log_e(1+x)$, prove that

$$\log_e 3=1+\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{5}\cdot\frac{1}{4^2}+\frac{1}{7}\cdot\frac{1}{4^3}+\ldots\ldots$$

$$\tfrac{1}{2}\log_e 10=\log_e 3+\frac{1}{19}+\frac{1}{3}\cdot\frac{1}{19^3}+\frac{1}{5}\cdot\frac{1}{19^5}+\ldots\ldots\text{"}$$

We have $\quad \log_e(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\&c.$

Changing the sign of x in this equation,

$$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\&c.$$

Subtracting this equation from the former,

$$\log_e(1+x)-\log_e(1-x),\text{ or }\log_e\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5},\ \&c.\right)\ .\ .\ (1)$$

Now let $\quad \dfrac{1+x}{1-x}=m$, whence $x=\dfrac{m-1}{m+1}$,

then $\quad \log_e m=2\left\{\dfrac{m-1}{m+1}+\dfrac{1}{3}\left(\dfrac{m-1}{m+1}\right)^3+\dfrac{1}{5}\left(\dfrac{m-1}{m+1}\right)^5+\ldots\right\}.$

Here make $m=3$, whence $\dfrac{m-1}{m+1}=\dfrac{2}{4}=\dfrac{1}{2}$,

then we get $$\log_e 3 = 2\left\{\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2^3}+\frac{1}{5}\cdot\frac{1}{2^5}+\ldots\right\}$$

$$=1+\frac{1}{3}\cdot\frac{1}{2^2}+\frac{1}{5}\cdot\frac{1}{2^4}+\frac{1}{7}\cdot\frac{1}{2^6}\cdot\ldots$$

$$=1+\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{5}\cdot\frac{1}{4^2}+\frac{1}{7}\cdot\frac{1}{4^3}+\ldots.$$

which is the first of the proposed formulæ.

Again, in formula (1) put $\frac{1+x}{1-x}=\frac{n+1}{n}$, whence $x=\frac{1}{2n+1}$; then we have

$$\log_e\frac{n+1}{n}=2\left\{\frac{1}{2n+1}+\frac{1}{3}\cdot\frac{1}{(2n+1)^3}+\frac{1}{5}\cdot\frac{1}{(2n+1)^5}+\ldots\right\}.$$

Here make $n=9$, and we have

$$\log_e\frac{10}{9}=2\left\{\frac{1}{19}+\frac{1}{3}\cdot\frac{1}{19^3}+\frac{1}{5}\cdot\frac{1}{19^5}+\ldots\right\}.$$

But $$\log\frac{10}{9}=\log_e 10-\log_e 9=\log_e 10-2\log_e 3,$$

$$\therefore \log_e 10-2\log_e 3=2\left\{\frac{1}{19}+\frac{1}{3}\cdot\frac{1}{19^3}+\frac{1}{5}\cdot\frac{1}{19^5}+\ldots\right\},$$

and $$\frac{1}{2}\log_e 10=\log_e 3+\frac{1}{19}+\frac{1}{3}\cdot\frac{1}{19^3}+\frac{1}{5}\cdot\frac{1}{19^5}+\ldots$$

—the second of the proposed formulæ.

2. "Explain what is meant by the modulus of a system of logarithms.
"By means of the formulæ in question (1), show that

$$\log_e 3 = 1{\cdot}098612;$$
$$\log_e 10 = 2{\cdot}302585.$$

"Also show that the modulus of the common system of logarithms is "·434294."

If N be any number, it can be proved that

$$\log_a N=\log_a e\,.\,\log_e N=\frac{\log_e N}{\log_e a},$$

whence we see that the logarithm of any number to the base a can be obtained by multiplying its Napierian (or hyperbolic) logarithm by the constant quantity $\log_a e$ or $\frac{1}{\log a}$. This quantity is called the *modulus* of the system of logarithms to the base a.

For the second part of the question, we have

$\frac{1}{4} = \cdot250{,}000{,}0$	$\frac{1}{3}\cdot\frac{1}{4} = \cdot083{,}333{,}3$
$\frac{1}{4^2} = \cdot062{,}500{,}0$	$\frac{1}{5}\cdot\frac{1}{4^2} = \cdot012{,}500{,}0$
$\frac{1}{4^3} = \cdot015{,}625{,}0$	$\frac{1}{7}\cdot\frac{1}{4^3} = \cdot002{,}232{,}1$
$\frac{1}{4^5} = \cdot003{,}906{,}3$	$\frac{1}{9}\cdot\frac{1}{4^5} = \cdot000{,}434{,}0$
$\frac{1}{4^6} = \cdot000{,}976{,}6$	$\frac{1}{11}\cdot\frac{1}{4^6} = \cdot000{,}088{,}8$
$\frac{1}{4^7} = \cdot000{,}244{,}2$	$\frac{1}{13}\cdot\frac{1}{4^7} = \cdot000{,}018{,}8$
$\frac{1}{4^8} = \cdot000{,}061{,}1$	$\frac{1}{15}\cdot\frac{1}{4^8} = \cdot000{,}004{,}1$
$\frac{1}{4^9} = \cdot000{,}015{,}3$	$\frac{1}{17}\cdot\frac{1}{4^9} = \cdot000{,}000{,}9$
$\frac{1}{4^{10}} = \cdot000{,}003{,}8$	$\frac{1}{19}\cdot\frac{1}{4^{10}} = \cdot000{,}000{,}2$
$\frac{1}{4^{11}} = \cdot000{,}001{,}0$	$\cdot098{,}612{,}2$

Hence, to six places of decimals, $\log_e 3 = 1\cdot098{,}612$.

For $\log_e 10$, we have

$\frac{1}{19} = \cdot052{,}631{,}6$	$\frac{1}{19} = \cdot052{,}631{,}6$
$\frac{1}{19^3} = \cdot000{,}145{,}8$	$\frac{1}{3}\cdot\frac{1}{19^3} = \cdot000{,}048{,}6$
$\frac{1}{19^5} = \cdot000{,}000{,}4$	$\frac{1}{5}\cdot\frac{1}{19^5} = \cdot000{,}000{,}1$
	$\cdot052{,}680{,}3$
Adding the value of $\log_e 3$ found above,	$\log_e 3 = 1\cdot098{,}612{,}2$
	$1\cdot151{,}292{,}5$
and multiplying by 2,	2
the second equation in question (1) gives the value of $\log_e 10$ required	$2\cdot302{,}585$

Here it may be noticed that in order to determine $\frac{1}{19^n}$, it is convenient to employ the binomial theorem.

$$\text{Thus, } \frac{1}{19^n} = \frac{1}{(20-1)^n} = \frac{1}{20^n}\left(1-\frac{1}{20}\right)^{-n}$$

$$= \frac{1}{20^n}\left\{1+\frac{n}{20}+\frac{n(n+1)}{1.2}\cdot\frac{1}{20^2}+\frac{n(n+1)(n+2)}{1.2.3}\cdot\frac{1}{20^3}+\&c.\right\}.$$

For example—

$$\frac{1}{19^3}=\frac{1}{20^3}\left\{1+\frac{3}{20}+\frac{3.4}{1.2}\cdot\frac{1}{20^2}+\frac{3.4.5}{1.2.3}\cdot\frac{1}{20^3}+\ldots\right\}$$

$$=\frac{1}{20^3}\left\{1+\frac{3}{20}+\frac{3}{200}+\frac{1}{800}+\ldots\right\}$$

$$=\frac{1}{20^3}\left\{\begin{array}{l}1+\cdot15\\ \;+\cdot015\\ \;+\cdot00125\end{array}\right\}$$

$$=\frac{1}{20^3}\times 1\cdot16625$$

$$= \quad \cdot000,145,8$$

The modulus of the common system of logarithms being $\frac{1}{\log_e 10}$, common division gives its value, $\frac{1}{2\cdot302,585}=\cdot434,294$.

Thus—

$$\begin{array}{r}2\cdot30\dot{2},\dot{5}\dot{8}\dot{5}\;)\,1\cdot000\;000\;00\,(\;\cdot434,294\\ 78\;966\;00\qquad\qquad\quad\\ 9\;888\;45\qquad\qquad\quad\\ 678\;11\qquad\qquad\quad\\ 217\;59\qquad\qquad\quad\\ 10\;36\qquad\qquad\quad\\ 1\;15\qquad\qquad\quad\end{array}$$

3. "When an event can happen in more ways than one, show that the "probability of its happening is equal to the sum of the probabilities in "respect of the different ways."

Vide Gray's Tables and Formulæ, p. 34.

4. "Determine (1) the probability that two persons now aged x and y "respectively will both die in the nth year from the present time; (2) the "probability that one or both of them will die in that year."

Using Jones's notation, the probability that x will live $n-1$ years is $p_{x.n-1}$, and the chance that he will live n years is $p_{x.n}$; therefore the chance that he will die in the nth year is $p_{x.n-1}-p_{x.n}$.

So the chance that y will die in the nth year is $p_{y.n-1}-p_{y.n}$, and the chance that *both* will die in that year is therefore

$$(p_{x.n-1}-p_{x.n})(p_{y.n-1}-p_{y.n}),$$

Again, the chance that x will *not* die in the nth year is $1-(p_{x.n-1}-p_{x.n})$, and the chance that y will not die in the nth year is $1-(p_{y.n-1}-p_{y.n})$. Hence the chance that neither x nor y will die in that year is

$$(1-p_{x.n-1}+p_{x.n})(1-p_{y.n-1}+p_{y.n});$$

and, consequently, the chance that one or both will die in that year is

$$1-(1-p_{x.n-1}+p_{x.n})(1-p_{y.n-1}+p_{y.n}).$$

5. "Find the present value of a sum of money due at the end of any "time, assuming the operation of compound interest."

Vide Jones on Annuities, p. 13, art. 21; *Colenso's Algebra*, p. 155, art. 198.

"Prove that the value of £1 due at the end of $p+q$ years=value of "£1 due in p years × value of £1 due in q years."

If i be the interest of £1 for 1 year—

then the value of £1 due in p years is $(1+i)^{-p}$,

" " q " $(1+i)^{-q}$,

and " " $p+q$ " $(1+i)^{-(p+q)}$.

Now, $(1+i)^{-p}\times(1+i)^{-q}=(1+i)^{-(p+q)}$,

which proves the property stated.

6. "Describe a practical method of calculating a table of the value of "£1 due at the end of any number of years from 1 to 100; and show how "the results may be employed to calculate a table of the values of annuities "certain."

See *Gray*, p. 22.

"*Ex.*—The value of £1 due at the end of 10 years, at 4 per cent., "is ·67556417: deduce the first ten terms of the value of an annuity "for any number of years."

Pursuing the method explained by Mr. Gray in the passage last referred to, the process will be as follows:—

$$
\begin{array}{lcrl}
(1+i)^{-10} & = & \cdot67556417 & \\
 & & 2702257 & = \cdot04\times(1+i)^{-10} \\
\hline
(1+i)^{-9} & = & \cdot70258674 & = (1+\cdot04)\,(1+i)^{-10} \\
 & & 2810347 & \\
\hline
(1+i)^{-8} & = & \cdot73069021 & \\
 & & 2922761 & \\
\hline
(1+i)^{-7} & = & \cdot75991782 & \\
 & & 3039671 & \\
\hline
(1+i)^{-6} & = & \cdot79031453 & \\
 & & 3161258 & \\
\hline
(1+i)^{-5} & = & \cdot82192711 & \\
 & & 3287708 & \\
\hline
(1+i)^{-4} & = & \cdot85480419 & \\
 & & 3419217 & \\
\hline
(1+i)^{-3} & = & \cdot88899636 & \\
 & & 3555985 & \\
\hline
(1+i)^{-2} & = & \cdot92455621 & \\
 & & 3698225 & \\
\hline
(1+i)^{-1} & = & \cdot96153846 & \\
 & & 3846154 & \\
\hline
(1+i)^{-0} & = & 1\cdot00000000 &
\end{array}
$$

Again, for the values of the annuities—

$$
\begin{array}{lrl}
(1+i)^{-1} = & \cdot 96153846 & = a_{\overline{1}|} \\
(1+i)^{-2} = & 92455621 & \\
\hline
 & 1\cdot 88609467 & = a_{\overline{2}|} \\
(1+i)^{-3} = & 88899636 & \\
\hline
 & 2\cdot 77509103 & = a_{\overline{3}|} \\
(1+i)^{-4} = & 85480419 & \\
\hline
 & 3\cdot 62989522 & = a_{\overline{4}|} \\
(1+i)^{-5} = & 82192711 & \\
\hline
 & 4\cdot 45182233 & = a_{\overline{5}|} \\
(1+i)^{-6} = & 79031453 & \\
\hline
 & 5\cdot 24213686 & = a_{\overline{6}|} \\
(1+i)^{-7} = & 75991782 & \\
\hline
 & 6\cdot 00205468 & = a_{\overline{7}|} \\
(1+i)^{-8} = & 73069021 & \\
\hline
 & 6\cdot 73274489 & = a_{\overline{8}|} \\
(1+i)^{-9} = & 70258674 & \\
\hline
 & 7\cdot 43533163 & = a_{\overline{9}|} \\
(1+i)^{-10} = & 67556417 & \\
\hline
 & 8\cdot 11089580 & = a_{\overline{10}|}
\end{array}
$$

The results in the former part of the process may be compared with those given in *Jones*, p. 80; and those in the latter part, with *Jones*, p. 92.

7. "Explain what is meant by a table of mortality, and state the "different methods in which the facts embodied in it may be exhibited."

For information on the subject of this question, reference may be made to De Morgan's *Essay on Probabilities*, pp. 161, 2.

8. "Give some account of the origin and relative merits of the tables "of mortality known as the Carlisle, Equitable, Experience, and English "Life Tables."

Vide Registrar-General's 5*th*, 6*th*, *and* 12*th Reports*, Dr. Farr's Appendices; *De Morgan on Probabilities*, p. 165; *Milne on Annuities*, p. 404; *Encyclopædia Britannica*, art. "Mortality"; *Morgan's Equitable Society Tables*; *Davies on Annuities*, chap. 3; *Jenkin Jones*, *New Rate of Mortality*, Introduction.

9. "State accurately what is meant by the expectation (or mean dura- "tion) of life at any age according to a given table of mortality."

On this point see *Gray*, art. 135 and 162–6.

"Prove the formula for calculating the chance of living a year at any "age from the expectations—

$$p_x = \frac{e_x - \frac{1}{2}}{e_{x+1} + \frac{1}{2}}."$$

It is proved (*Jones*, art. 122) that

$$e_x = \tfrac{1}{2} + \frac{l_{x+1} + l_{x+2} + \ldots\ldots + l_{x+z}}{l_x}.$$

Thus then, $$e_x = \tfrac{1}{2} + \frac{l_{x+1}}{l_x} \cdot \frac{l_{x+1} + l_{x+2} + \ldots\ldots + l_{x+z}}{l_{x+1}}$$

$$= \tfrac{1}{2} + \frac{l_{x+1}}{l_x}\left(1 + \frac{l_{x+2} + l_{x+3} + \ldots\ldots + l_{x+z}}{l_{x+1}}\right).$$

Now, $\frac{l_{x+1}}{l_x} = p_x$, and $e_{x+1} = \frac{1}{2} + \frac{l_{x+2} + l_{x+3} + \ldots\ldots + l_{x+z}}{l_{x+1}}$.

Hence by substitution $$e_x = \tfrac{1}{2} + p_x(\tfrac{1}{2} + e_{x+1}),$$

whence we get $$p_x = \frac{e_x - \frac{1}{2}}{e_{x+1} + \frac{1}{2}}.$$

See also *Gray*, art. 152, 3.

10. "Prove that if $e_{x-1} = e_x = e_{x+1}$, then will $p_{x-1} = p_x$; but that if "$p_{x-1} = p_x = p_{x+1}$, e_{x-1} will not be equal to e_x unless $e_{x+2} = \frac{1}{2} \cdot \frac{1 + p_x}{1 - p_x}$."

By the formula in the last question,

$$p_x = \frac{e_x - \frac{1}{2}}{e_{x+1} + \frac{1}{2}}, \quad p_{x-1} = \frac{e_{x-1} - \frac{1}{2}}{e_x + \frac{1}{2}}, \quad p_{x+1} = \frac{e_{x+1} - \frac{1}{2}}{e_{x+2} + \frac{1}{2}}.$$

If, then, $e_{x-1} = e_x = e_{x+1}$, it is clear that $p_x = p_{x-1} = \frac{e_x - \frac{1}{2}}{e_x + \frac{1}{2}}$.

Again, if $$p_{x-1} = p_x = p_{x+1},$$

we have $$p_x = \frac{e_{x-1} - \frac{1}{2}}{e_x + \frac{1}{2}} = \frac{e_x - \frac{1}{2}}{e_{x+1} + \frac{1}{2}} = \frac{e_{x+1} - \frac{1}{2}}{e_{x+2} + \frac{1}{2}};$$

whence $$e_{x+1} = \tfrac{1}{2} + p_x(e_{x+2} + \tfrac{1}{2}),$$

$$e_x = \tfrac{1}{2} + p_x(e_{x+1} + \tfrac{1}{2}),$$

$$e_{x-1} = \tfrac{1}{2} + p_x(e_x + \tfrac{1}{2}).$$

From these equations, it appears that e_{x-1} will not be equal to e_x unless $e_x = e_{x+1} = e_{x+2}$, and then the first of them gives

$$e_{x+2} = \tfrac{1}{2} + p_x(e_{x+2} + \tfrac{1}{2});$$

from which it follows, that

$$e_{x+2} = \tfrac{1}{2} \cdot \frac{1 + p_x}{1 - p_x}.$$

11. "The value of an annuity on the life of a person of a given age is "frequently supposed to be equal to that of an annuity certain for a number "of years equal to the mean duration of life at that age. Explain why this "is not the case, and state upon what hypothesis it would be true."

See *Jones*, art. 127, p. 122; *De Morgan on Probabilities*, p. 188.

The following mathematical proof will probably be considered by some persons more satisfactory than the reasonings in the passages above referred to.

The value of an annuity on the life x is

$$\frac{l_{x+1}}{l_x}.r+\frac{l_{x+2}}{l_x}.r^2+\ldots\ldots+\frac{l_{x+z}}{l_x}.r^z,$$

or,
$$p_1r+p_2r^2+\ldots\ldots+p_zr^z, \text{ suppose;}$$

where it must be noticed that all the quantities $p_1, p_2 \ldots\ldots p_z$, are from the nature of the case, less than unity. Now let n be the number of complete years in the mean duration of life at the age x, then the value of an annuity certain for n years is

$$r+r^2+r^3+\ldots\ldots+r^n,$$

and we have

$$n+\text{a fraction}=\tfrac{1}{2}+\frac{l_{x+1}+l_{x+2}+\ldots\ldots+l_{x+z}}{l_x},$$

or
$$n+\delta=\tfrac{1}{2}+p_1+p_2+\ldots\ldots+p_z.$$

Then the value of the annuity on the life x will be greater or less than the annuity certain for n years, according as

$$p_1r+p_2r^2+\ldots\ldots+p_zr^z > \text{or} < r+r^2+\ldots\ldots+r^n,$$

and therefore according as

$$p_{n+1}r^{n+1}+\ldots\ldots+p_zr^z > < (1-p_1)r+(1-p_2)r^2+\ldots\ldots+(1-p_n)r^n.$$

Now, since $r<1$, the first member of the inequality must be

$$< p_{n+1}r^{n+1}+p_{n+2}.r^{n+1}+\ldots\ldots+p_z.r^{n+1},$$
$$< (p_{n+1}+p_{n+2}+\ldots\ldots+p^z)r^{n+1};$$

and the second member of the inequality must be

$$>(1-p_1+1-p_2+\ldots\ldots+1-p_n)r^n,$$
$$>(n-p_1-p_2-\ldots\ldots-p_n)r^n,$$
$$>(p_{n+1}+p_{n+2}+\ldots\ldots+p_z+\tfrac{1}{2}-\delta)r^n.$$

If, then, we suppose δ to be not greater than $\frac{1}{2}$, or if we take n to be the *nearest* integer to the number expressing the mean duration of life, instead of the integer next less than it, we shall have

$$(p_{n+1}+p_{n+2}+\ldots\ldots+p_z)r^{n+1}<(p_{n+1}+p_{n+2}+\ldots\ldots+p_z+\tfrac{1}{2}-\delta)r^n,$$

$$\therefore p_{n+1}r^{n+1}+p_{n+2}r^{n+2}+\ldots\ldots+p_zr^z$$
$$<(1-p_1)r+(1-p_2)r^2+\ldots\ldots+(1-p_n)r^n,$$

whence the value of the annuity on the life x is always less than that of the annuity certain for the number of years indicated by the nearest integer to the number expressing the mean duration of life.

12. "If B represents the present value of a benefit of £1 upon a given "life (x), B_1 the same upon a life one year older ($x+1$), p the probability "of a payment of B being received in the first year, and Π the probability "of (x) surviving a year, prove that

$$\log B = \log v\,\Pi + \log\left(\frac{p}{\Pi} + B_1\right).\text{"}$$

For demonstration of this formula, see *Gray*, art. 193, 4.

13. "Describe the process of calculating a table of annuities by means "of the formula $a_{m-1}=(1+a_m)p_{m-1,1}.r$; and show that this formula is a "particular case of the one in the last question."

For the first part of this question, see *Jones*, art. 118, 9; for the second part, see *Gray*, art. 202.

14. "Explain fully the construction and use of columns D, N, and M, "in the columnar method of calculating the values of annuities and assur- "ances; also state the superior advantages this method possesses over the "old one."

See *Jones*, arts. 112–6; *Gray*, arts. 296–311; *De Morgan*, *Companion to the Almanac* for 1840.

15. "Prove that the value of an annuity of £1 during the joint lives of "x and y, and for t years afterwards, should x survive so long, is

$$a_x - \frac{D_{x+t}}{D_x}(a_{x+t} - a_{x+t.y}).\text{"}$$

This problem has been investigated by Jones (art. 161), who has given, for the annuity in question, the formula—

$$a_{x_{\overline{t}|}} + \frac{a_{x.y-t_{\overline{|}t}}}{p_{y-t,t}}.$$

This may be shown, as follows, to be equivalent to the formula given above.

We have
$$a_{x_{\overline{t}|}} = a_x - p_{x.t}r^t.\,a_{x+t}$$
$$= a_x - \frac{D_{x+t}}{D_x}.\,a_{x+t}.$$

Again,
$$a_{x.y-t_{\overline{|}t}} = p_{(x.y-t)t}.\,r^t a_{x+t.y},$$

and, consequently,

$$\frac{a_{x.y-t_{\overline{|}t}}}{p_{y-t.t}} = \frac{p_{(x.y-t)t}}{p_{y-t.t}}.\,r^t.\,a_{x+t.y}$$
$$= p_{x.t}r^t.\,a_{x+t.y}$$
$$= \frac{D_{x+t}}{D_x}.\,a_{x+t.y}.$$

Substituting these values in (1), we get the formula stated in the question.

On this point reference may be made to the *Assurance Magazine*, vol. vii., p. 239; or the formula may be proved by a process of general reasoning similar to that employed in the *Assurance Magazine*, vol. vii., p. 174.

16. "Give an expression for the annual premium for a contingent "annuity to commence at the death of A and to continue as long as either "B or C is living."

It will be noticed that it is only required here that the formula should be stated, not proved.

If x, y, z, are the ages of the lives at the present time, the annual premium will be

$$\frac{a_y + a_z - a_{y.z} - a_{x.y} - a_{x.z} + a_{x.y.z}}{1 + a_{x.y} + a_{x.z} - a_{x.y.z}}.$$

See *Jones*, art. 165.

17. "Find, in a convenient form for computation, the single and annual "premiums for an annuity to commence at the death of y and continue pay-"able during the remainder of x's life, but to be payable only if y dies within "t years."

If, to the annuity described in the question, we add a reversionary annuity to x after y, payable only in case both x and y survive t years, the sum will be an ordinary reversionary annuity to x after y, the formula for which is $a_x - a_{xy}$. But the formula for the deferred reversionary annuity contingent on x and y both surviving t years is easily seen to be

$$p_{(x.y)t} . r^t . \{a_{x+t} - a_{x+t.y+t}\},$$

$$\text{or } \frac{D_{x+t.y+t}}{D_{x.y}} \{a_{x+t} - a_{x+t.y+t}\}.$$

Hence the single premium for the proposed annuity will be

$$a_x - a_{xy} - \frac{D_{x+t.y+t}}{D_{x.y}} \{a_{x+t} - a_{x+t.y+t}\}.$$

Also, the divisor for the annual premium will be $1 +$ an annuity for $t-1$ years on the joint lives of x and y,

$$= 1 + a_{x.y_{\overline{t-1}|}} = 1 + a_{x.y} - \frac{D_{x+t.y+t}}{D_{x.y}} \{1 + a_{x+t.y+t}\}.$$

18. "Prove that the single premium for an assurance payable on a life "now aged n years attaining the age of $n+t$, or dying previously, may be "represented by the formula (Jones's notation), $\frac{1 - ia_{n_{\overline{t-1}|}}}{1+i}$; and show that "this formula is equivalent to $\frac{M_n - M_{n+t} + D_{n+t}}{D_n}$."

The annual interest on £1 being i, it is obvious that £1 is equal in value to an annuity of i for any term and a capital sum of $1+i$ at the end

of that term,—the sum $1+i$ being payable a year after the last payment of the annuity. Now let that term be $t-1$ years contingent on the existence of the life n, then

$1=$an annuity of i for $t-1$ years on the life n
$+(1+i)$ receivable at the end of the year in which n dies, or at the end of n years, whichever first happens;

or, $$1=ia_{n\overline{t-1}|}+(1+i)\mathrm{R}$$

putting R for the value of the assurance in question.

Hence $\mathrm{R}=\dfrac{1-ia_{n\overline{t-1}|}}{1+i}$, as stated in the question.

Next to prove that this formula is equivalent to $\dfrac{\mathrm{M}_n-\mathrm{M}_{n+t}+\mathrm{D}_{n+t}}{\mathrm{D}_n}$, which is easily seen to be another expression for the same assurance. (*V. Jones*, p. 212.)

We have $$a_{n\overline{t-1}|}=\frac{\mathrm{N}_n-\mathrm{N}_{n+t-1}}{\mathrm{D}_n}.$$

Hence $$\frac{1-ia_{n\overline{t-1}|}}{1+i}=r-\frac{d\mathrm{N}_n-d\mathrm{N}_{n+t-1}}{\mathrm{D}_n}$$
$$=\frac{r\mathrm{D}_n-d\mathrm{N}_n+d\mathrm{N}_{n+t-1}}{\mathrm{D}_n}.$$

But, by the construction of the column M, we have the relations (see *Gray*, art. 305, p. 118),

$$\mathrm{M}_n=\mathrm{D}_n-d\mathrm{N}_{n-1}$$
$$=\mathrm{D}_n-d(\mathrm{D}_n+\mathrm{N}_n)$$
$$=r\mathrm{D}_n-d\mathrm{N}_n;$$

and $$\mathrm{M}_{n+t}=\mathrm{D}_{n+t}-d\mathrm{N}_{n+t-1},$$

whence $$d\mathrm{N}_{n+t-1}=\mathrm{D}_{n+t}-\mathrm{M}_{n+t}.$$

We thus have
$$r\mathrm{D}_n-d\mathrm{N}_n+d\mathrm{N}_{n+t-1}=\mathrm{M}_n+\mathrm{D}_{n+t}-\mathrm{M}_{n+t},$$

which proves the equivalence of the two expressions.

19. "Give an expression for the single premium for an assurance on "the life of A provided he die after B."

The expression required is
$$\mathrm{A}_x-\mathrm{A}_{\underset{1}{x.y}},$$

x and y being the ages of A and B. See *Jones*, art. 216.

20. "Prove that $\mathrm{A}_{\underset{1}{(x.y)}\rceil t}=r^t.p_{(x.y)t}.\mathrm{A}_{\underset{1}{x+t.y+t}}$."

Adopting Jones's notation, let $q_{\underset{1}{(x.y)}n}$ denote the chance of x dying before y in the nth year from the present time; then

$$\mathrm{A}_{\underset{1}{(x.y)}\rceil t}=r^{t+1}q_{\underset{1}{(x.y)}t+1}+r^{t+2}q_{\underset{1}{(x.y)}t+2}+\&c.$$

But a little consideration shows that $q_{\underset{1}{(x.y)}t+n}$ is compounded of two chances—first, the chance that x and y will both live t years; and, secondly, the chance that, having survived t years, x will die before y in the nth year from that time, which is the same thing as the chance that a life $x+t$ will die before $y+t$ in the nth year. Hence—

$$q_{\underset{1}{(x.y)}t+n}=p_{(x.y)t}\cdot q_{\underset{1}{(x+t.y+t)}n}\,.$$

Making n in this equation successively equal to 1, 2 . . . . , and substituting above, we have

$$\begin{aligned} A_{\underset{1}{(x.y)}\rceil t} &= r^{t+1}p_{(x.y)t}\cdot q_{\underset{1}{(x+t.y+t)}1}+r^{t+2}p_{(x.y)t}\cdot q_{\underset{1}{(x+t.y+t)}2}+\&c. \\ &= r^{t}p_{(x.y)t}\{rq_{\underset{1}{(x+t.y+t)}1}+r^{2}q_{\underset{1}{(x+t.y+t)}2}+\&c.\} \\ &= r^{t}p_{(x.y)t}\cdot A_{\underset{1}{x+t.y+t}}\,. \end{aligned}$$

On this subject see *Gray*, art. 259, 60.

21. "Determine the value of an assurance on a life of 30, payable at "the age of 60 or previous death, commencing at £a, increased by £p at "the end of 5 and 10 years respectively, and thereafter increasing by £q "*per annum.*"

The value of the assurance of £a, payable at 60 or death, is

$$a\,.\,\frac{M_{30}-M_{60}+D_{60}}{D_{30}}\,.$$

The values of the deferred assurances of £p, payable at 60 or death, are

$$p\,.\,\frac{M_{35}-M_{60}+D_{60}}{D_{30}},\quad p\,.\,\frac{M_{40}-M_{60}+D_{60}}{D_{30}}\,.$$

The value of the increasing assurance of £q *per annum* after 10 years will be

$$\begin{aligned} & q\frac{M_{41}-M_{60}+D_{60}}{D_{30}}+q\frac{M_{42}-M_{60}+D_{60}}{D_{30}}+\ .\ .\ .\ . \\ & \qquad +q\frac{M_{59}-M_{60}+D_{60}}{D_{30}}+q\frac{M_{60}-M_{60}+D_{60}}{D_{30}} \\ & = q\frac{M_{41}+M_{42}+\ .\ .\ .\ .\ +M_{60}}{D_{30}}+20q\frac{D_{60}-M_{60}}{D_{30}} \\ & = q\frac{R_{41}-R_{61}}{D_{30}}+20q\frac{D_{60}-M_{60}}{D_{30}}\,. \end{aligned}$$

Adding the different quantities together, the value of the assurance proposed is

$$\frac{aM_{30}+p(M_{35}+M_{40})+q(R_{41}-R_{61})+(a+2p+20q)(D_{60}-M_{60})}{D_{30}}\,.$$

22. "Prove the formulæ for the value of a policy of £1—

$$(1)\ 1-\frac{1+a_{m+n}}{1+a_m},\quad (2)\ (\mathrm{P}_{m+n}-\mathrm{P}_m)(1+a_{m+n}),$$

$$(3)\ 1-(d+\mathrm{P}_m)(1+a_{m+n})."$$

For (1) and (2), see *Jones*, art. 254.

The formula (3) is deduced from (1) by means of the equation (*Jones*, art. 191)—

$$\mathrm{P}_m=\frac{1}{1+a_m}-(1-r)=\frac{1}{1+a_m}-d,$$

whence
$$\mathrm{P}_m+d=\frac{1}{1+a_m}.$$

"Show that the same forms apply when the policy is on the joint "duration of two lives."

In order to prove this, it is only necessary to notice that all the formulæ employed in the proof of the above expressions are equally true whether the policy is on one life or on the first of two lives; and, therefore, those expressions must also be equally applicable in both cases.

23. "Find the value of an annuity on two successive lives, x and y, "of which the second is to be nominated at the death of the first, and is "supposed to be then y years of age."

See *Jones*, art. 238; or *Assurance Magazine*, vol. ii., p. 5.

"Also show that if I represent the value of a perpetuity of £1, the "value of the annuity will be equal to $\mathrm{I}-(1+\mathrm{I})\mathrm{A}_x\mathrm{A}_y$."

If to the annuity on the successive lives be added a perpetuity, to commence on the death of y, and the first payment to be made at the end of the year in which y dies, the sum will be equal to a perpetuity from the present time$=$I. Now the value of the deferred perpetuity-due will be that of an assurance of $1+\mathrm{I}$ on death of y, the value of which will be $(1+\mathrm{I})\mathrm{A}_y$ at the time y is nominated, *i.e.* at the end of the year in which x dies, and therefore, equal to $(1+\mathrm{I})\mathrm{A}_x\mathrm{A}_y$ at the present time. Hence the value of the annuity on the succession is, as stated in the question, $\mathrm{I}-(1+\mathrm{I})\mathrm{A}_x\mathrm{A}_y$.

The formula given by Jones is $a_x+\dfrac{1-ia_x}{1+i}(1+a_y)$, which may be shown to be equivalent to the above, as follows:—

We have
$$\frac{1-ia_x}{1+i}=\mathrm{A}_x.$$

Hence $1-ia_x=(1+i)\mathrm{A}_x$; or, since $\frac{1}{i}=\mathrm{I}$, $\mathrm{I}-a_x=(1+\mathrm{I})\mathrm{A}_x$, and

$$a_x=\mathrm{I}-(1+\mathrm{I})\mathrm{A}_x.$$

So $a_y=\mathrm{I}-(1+\mathrm{I})\mathrm{A}_y$, and

$$1+a_y=1+\mathrm{I}-(1+\mathrm{I})\mathrm{A}_y=(1+\mathrm{I})(1-\mathrm{A}_y).$$

Substituting these values in Jones's formula, it becomes

$$\mathrm{I}-(1+\mathrm{I})\mathrm{A}_x+\mathrm{A}_x(1+\mathrm{I})(1-\mathrm{A}_y)=\mathrm{I}-(1+\mathrm{I})\mathrm{A}_x\mathrm{A}_y.$$

24. "Find the single premium for an annuity to x after the death of y, "with the condition that the premium is to be returned if x die before y."

Let P be the premium required, then the benefit to be insured is an annuity to x after y, or a_x-a_{xy}, and an insurance of P if x die before y, or $\mathrm{P}.\mathrm{A}_{\underset{1}{x}.y}$; and these together must be equal to P.

Thus, then, $$a_x-a_{xy}+\mathrm{PA}_{\underset{1}{x}.y}=\mathrm{P},$$

whence $$\mathrm{P}(1-\mathrm{A}_{\underset{1}{x}.y})=a_x-a_{xy},$$

and $$\mathrm{P}=\frac{a_x-a_{xy}}{1-\mathrm{A}_{\underset{1}{x}.y}}.$$

25. "A, aged x years, is entitled to the interest of £1 for life. If A "die within t years, the interest is payable to B or his representatives "till the expiration of the t years; when C, if living, is entitled to the "capital. If, however, C, now aged y years, die either before A or within "the t years, the capital reverts to B or his representatives. Determine "the values of B's and of C's interests."

Let P and Q denote the values of B's and C's interests, then B's interest during t years, or $\mathrm{P}_{t\rceil}$, consists of two parts, first his interest in the income (£i per annum) which he will receive after the death of A and to the end of the t years if C live so long; secondly, his chance of receiving the capital by the death of both x and y within the t years. In this case it must be remembered that he will be entitled to a year's income in addition to the capital, at the end of the year in which the last survivor of x and y dies, so that this second part is equal to an insurance of $1+i$ on the death of the last survivor of x and y provided that death take place within t years.

Hence $$\mathrm{P}_{t\rceil}=i(a_{y_{t}\rceil}-a_{x.y_{t}\rceil})+(1+i)(\mathrm{A}_{x_{t}\rceil}+\mathrm{A}_{y_{t}\rceil}-\mathrm{A}_{x.y_{t}\rceil}).$$

Again, B's interest after the expiration of the t years consists of the chance of his becoming entitled to the capital and one year's income, on the death of A after the expiration of t years, C having died before A.

Thus $$\mathrm{P}_{\rceil t}=(1+i)\mathrm{A}_{\underset{2}{(x.y)}\rceil t}=(1+i)\{\mathrm{A}_{x\rceil t}-\mathrm{A}_{\underset{1}{(x.y)}\rceil t}\}.$$

Adding the two parts of B's interest together, we have

$$\begin{aligned}\mathrm{P}&=\mathrm{P}_{t\rceil}+\mathrm{P}_{\rceil t}\\&=i(a_{y_{t}\rceil}-a_{x.y_{t}\rceil})+(1+i)\{\mathrm{A}_{x_{t}\rceil}+\mathrm{A}_{y_{t}\rceil}-\mathrm{A}_{x.y_{t}\rceil}+\mathrm{A}_{x\rceil t}-\mathrm{A}_{\underset{1}{x}.y\rceil t}\}\\&=i(a_{y_{t}\rceil}-a_{x.y_{t}\rceil})+(1+i)\{\mathrm{A}_x+\mathrm{A}_{y_{t}\rceil}-\mathrm{A}_{x.y_{t}\rceil}-\mathrm{A}_{\underset{1}{x}.y\rceil t}\}.\quad.\;.\;.\;(1)\end{aligned}$$

Now it is proved (*Jones*, art. 197) that

$$(1+i)A_{y_{t}\rceil}=1-r^{t}p_{y.t}-i\,a_{y_{t}\rceil},$$

whence
$$i\,a_{y_{t}\rceil}+(1+i)A_{y_{t}\rceil}=1-r^{t}p_{y.t}.\text{*}$$

So also
$$i\,a_{x.y_{t}\rceil}+(1+i)A_{x.y_{t}\rceil}=1-r^{t}p_{(x.y)t}.$$

Substituting now in equation (1) we get

$$P=(1+i)(A_x-A_{\underset{1}{x}.y\rceil_t})-r^{t}p_{y.t}+r^{t}p_{(x.y)t}$$

$$=(1+i)(A_x-A_{\underset{1}{x}.y\rceil_t})-r^{t}p_{y.t}(1-p_{x.t}).$$

For the value of C's interest, we notice that during the t years it is zero, and after t years it is equal to an assurance of $1+i$ at the death of x, provided that take place after the t years and y be then surviving, and to an endowment of £1 at the end of t years contingent on y being then alive and x previously dead. It is thus equal to

$$(1+i)A_{\underset{1}{x}.y\rceil_t}+r^{t}p_{y.t}(1-p\quad).$$

Adding together the two values, we get

$$P+Q=(1+i)A_x,$$

as we evidently ought, since the sum of the interests of B and C is equal to a reversion to the capital and one year's income on the death of A.

DR. FARR'S HEALTHY LIFE TABLE.

To the Editor of the Assurance Magazine.

SIR,—I forward to you D, N, and S, computed from Dr. Farr's Healthy Life Table, to enable Actuaries to compare it with other tables.

I have not added any numerical comparisons; but my own opinion, at present, is, that Dr. Farr's Table is about as much an advance on the Carlisle Table at that was on the Northampton. But whether the Assurance Offices will adopt it, is another question. As regards the law of mortality, I may mention that there is a remarkable table by Brune, derived from the Prussian returns made during a long course of years, which, on account of the well-known care exercised, well deserves attention.

I am, Sir, &c., &c.,

WM. DAVIS.

22, *Grove Place, St. John's Wood.*
May 13*th*, 1861.

* This equation may be established as follows:—The yearly interest of £1 being £i, it is obvious that £1 is equal in value to an annuity of £i for any term and an assurance of $1+i$ at the end of that term: so £1 is equal in value to a temporary annuity of £i on the life of y, together with a temporary assurance of $1+i$ on y, together with an endowment of £1 at end of t years if y is then alive; which is exactly what is expressed by the equation in question.

Dr. Farr's Healthy English Life Table (3 *per Cent.*).

Yrs.	D.	N.	S.	Yrs.	D.	N.	S.
0	100000·00	2132532·64	47235926·60	54	10985·94	149055·44	1576768·50
1	87092·22	2045440·42	45103394·44	55	10508·93	138546·51	1427713·06
2	81723·05	1963717·37	43057954·50	56	10046·20	128500·31	1289166·55
3	77617·74	1886099·63	41094237·61	57	9597·25	118903·06	1160666·24
4	74197·54	1811902·09	39208138·46	58	9156·69	109746·37	1041763·18
5	71129·85	1740772·24	37396236·85	59	8722·896	101023·47	932016·81
6	68348·75	1672423·49	35655465·09	60	8292·31	92731·16	830993·34
7	65803·49	1606620·00	33983042·08	61	7865·72	84865·44	738262·18
8	63448·77	1543171·23	32376422·56	62	7443·82	77421·62	653396·74
9	61248·94	1481922·29	30833251·81	63	7028·44	70393·18	675975·12
10	59174·06	1422748·23	29351330·00	64	6618·85	63774·33	505581·94
11	57199·86	1365548·37	27928582·25	65	6216·71	57557·62	441807·61
12	55306·59	1310241·78	26563034·36	66	5822·27	51735·35	384249·99
13	53478·51	1256763·27	25252793·06	67	5435·88	46299·47	332514·64
14	51704·04	1205059·23	23996030·27	68	5058·07	41241·40	286215·17
15	48973·66	1156085·57	22790971·52	69	4688·96	36552·44	244973·77
16	48285·45	1107800·12	21634886·43	70	4329·21	32223·23	208421·33
17	46624·99	1061175·13	20527086·79	71	3979·35	28243·88	176198·10
18	45002·35	1016172·78	19465912·14	72	3640·11	24603·77	147954·22
19	43412·45	972760·33	18449739·84	73	3311·90	21291·87	123350·45
20	41857·88	930902·45	17476979·99	74	2996·21	18295·66	102058·58
21	40341·99	890560·46	16546078·02	75	2693·23	15602·43	83762·92
22	38873·69	851686·77	15655517·56	76	2404·19	13198·24	68160·49
23	37452·11	814234·66	14803830·79	77	2130·13	11068·11	54962·25
24	36077·53	778157·13	13989596·13	78	1872·07	9196·04	43894·14
25	34748·19	743408·94	13211439·00	79	1630·62	7565·42	34698·10
26	33463·25	709945·69	12468030·06	80	1406·91	6158·510	27132·68
27	32222·73	677722·96	11758084·37	81	1201·44	4957·070	20974·17
28	31025·00	646697·96	11080361·41	82	1014·70	3942·373	16017·10
29	29868·87	616829·09	10433663·45	83	846·865	3095·508	12074·727
30	28753·36	588075·73	9816834·36	84	697·703	2397·805	8979·219
31	27644·31	560431·42	9228758·63	85	567·281	1830·672	6581·414
32	26637·98	533793·44	8668327·21	86	454·281	1376·391	4750·742
33	25636·26	508157·18	8134533·77	87	358·219	1018·172	3374·351
34	24669·58	483487·60	7626376·59	88	277·837	740·335	2356·179
35	23737·40	459750·20	7142888·99	89	211·755	528·580	1615·844
36	22838·03	436912·17	6683138·79	90	158·387	370·193	1087·264
37	21970·50	414941·67	6246226·62	91	116·094	254·099	717·071
38	21132·86	393808·81	5831284·95	92	83·448	170·651	462·972
39	20324·68	373484·13	5437476·14	93	58·6183	112·033	292·322
40	19544·94	353939·19	5063992·01	94	40·1978	71·835	180·289
41	18791·63	335147·56	4710052·82	95	26·9028	44·932	108·454
42	18064·28	317083·28	4374905·26	96	17·5690	27·363	63·522
43	17361·38	299721·90	4057821·98	97	11·1440	16·2193	36·159
44	16681·93	283039·97	3758100·08	98	6·9002	9·3191	19·9396
45	16025·51	267014·46	3475060·11	99	4·12673	5·19237	10·6205
46	15390·32	251624·14	3208045·65	100	2·39351	2·79886	5·42813
47	14776·06	236848·08	2956421·51	101	1·36397	1·43489	2·62927
48	14181·35	222666·72	2719573·43	102	0·73569	0·69920	1·19438
49	13605·73	209061·00	2496906·70	103	0·38094	0·31826	0·49518
50	13048·41	196012·59	2287845·70	104	0·18492	0·13334	0·17692
51	12508·46	183504·13	2091833·11	105	0·08977	0·04358	0·04358
52	11985·03	171519·10	1908328·98	106	0·04358	0·00000	0·00000
53	11477·72	160041·38	1736809·88				

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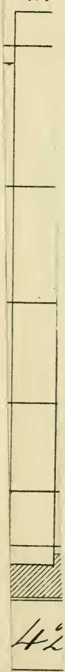
2.



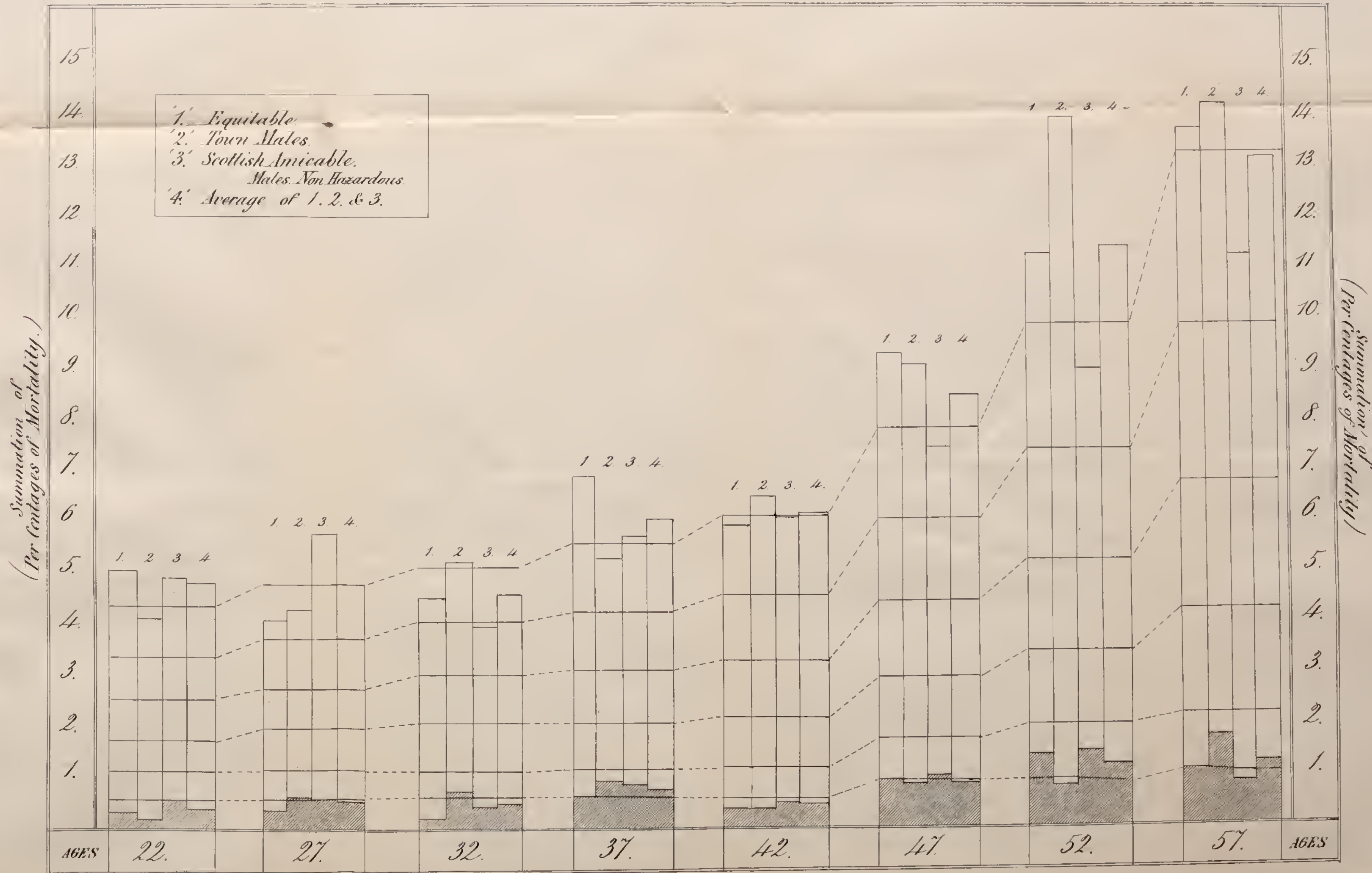
42

DIAGRAM No. 1.

The darkened portion at foot of the Diagram shows the per Centages of Mortality for the first year of the Assurances.

The height of the columns shows the summation of the per Centages for the first 6 years. And

The six straight lines drawn across represent the Summation of the Deduced per Centages of Mortality for the first six years respectively.



A

NERALLY DEDUCED.

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wing differences

Values, 1844

Table II

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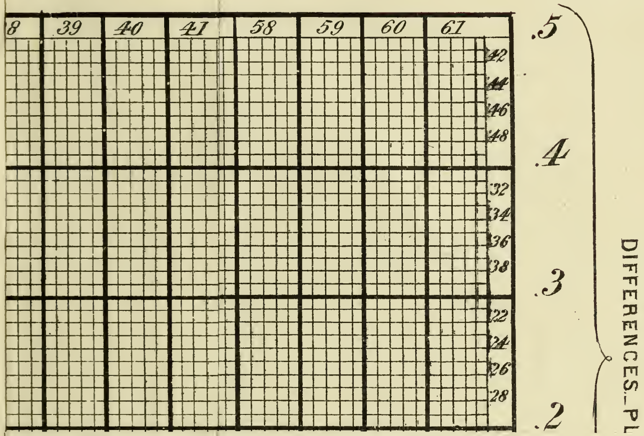
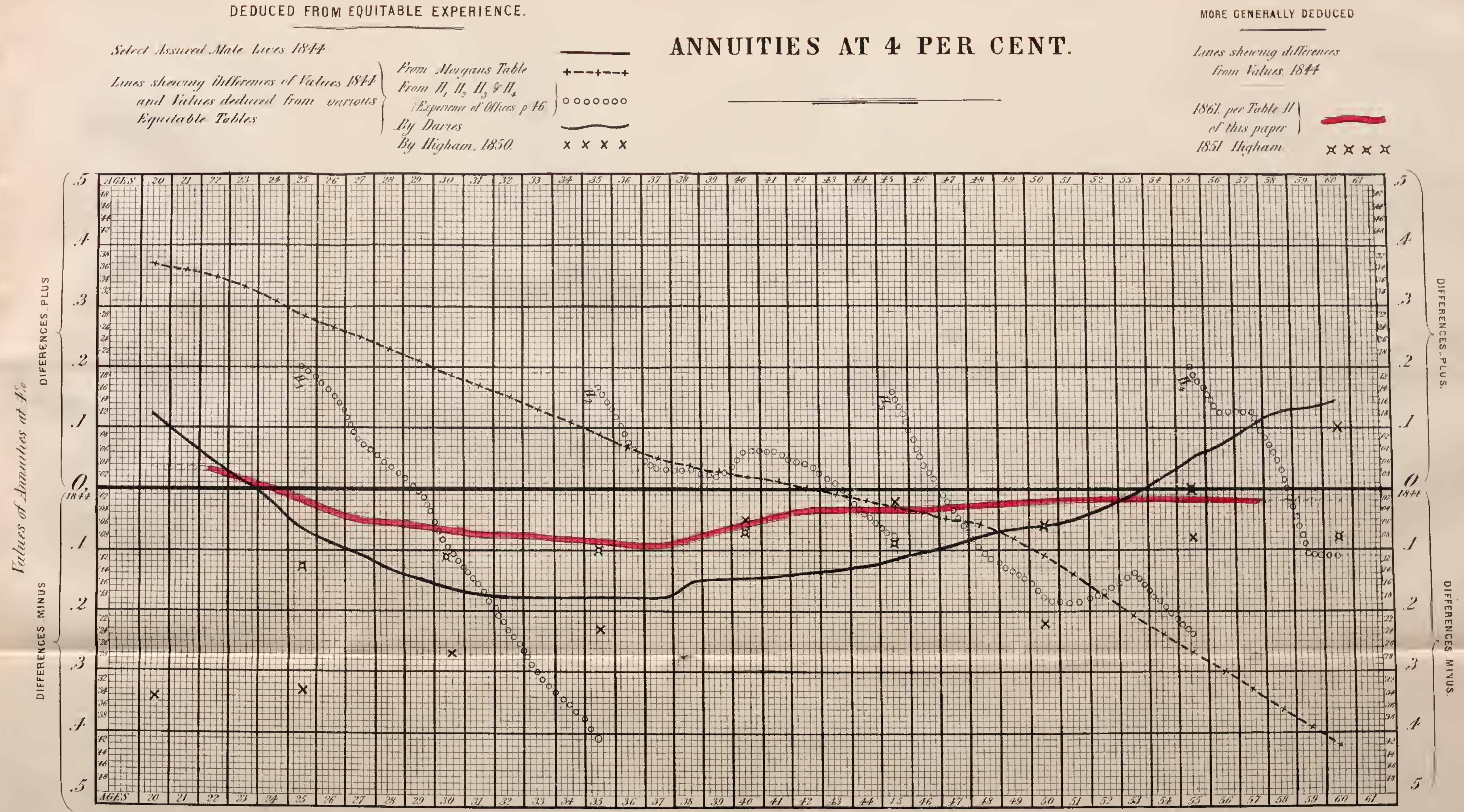


DIAGRAM No. 2.



DEDUCED FROM EQUITABLE EXPERIENCE.
ANNUITIES AT 4 PER CENT.
MORE GENERALLY DEDUCED
Select Assured Male Lives. 1844
Lines shewing Differences of Values 1844 and Values deduced from various Equitable Tables
From Morgans Table
From H_1 H_2 H_3 & H_4 (Experience of Offices p 46)
By Davies
By Higham, 1850.
Lines shewing differences from Values. 1844
1861. per Table II of this paper
1851 Higham
AGES
DIFFERENCES. PLUS
DIFFERENCES. MINUS
Values of Annuities at 4%
1844

THE

ASSURANCE MAGAZINE,

AND

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

Observations submitted by William Spens, *Manager of the Scottish Amicable Life Assurance Society, in presenting to the Institute Tables of the Mortality Experience of that Society from* 1826 *to* 1860.

[Read before the Institute 25th November, 1861, and printed by order of the Council.]

As stated in the introduction to the Tables, they relate to 10,255 persons, as under—

	Entered.	Discontinued.	Died.	Existing.
1. Non-hazardous—Males . .	8,596	2,550	632	5,414
2. Non-hazardous—Females . . .	890	247	63	580
3. Hazardous—Both sexes (exclusive of West Indies)	431	132	47	252
4. Hazardous—Both sexes (West Indies)	338	129	31	178
Total . .	10,255	3,058	773	6,424

And they embrace the experience of the first thirty-four years of the Society—reckoning, as was found most suitable, the first year's experience as the experience of the persons entering in the first year for one year from their respective entries, and so on. In this way it will be found that the experience does not include any of the persons entering in 1860, and none of the deaths in 1860 of persons who had been assured for more than a year from the time of payment of their annual premium in 1859. These deaths,

and the entries of 1860, fall, according to the principle adopted, to be included in the experience of the thirty-fifth year of the Society.

The sums of the numbers of lives exposed to risk during each year of age amount to—

For Non-hazardous—Males	48,348·0
,, Non-hazardous—Females	4,518·0
,, Hazardous (both sexes), exclusive of West Indies	1,964·0
,, Hazardous (both sexes), West Indies . .	1,471·5
Total . . .	56,301·5

The average duration of the lives being—

For Non-hazardous—Males	5·62
,, Non-hazardous—Females	5·07
,, Hazardous (both sexes), exclusive of West Indies .	4·55
,, Hazardous (both sexes), West Indies . . .	4·35
,, Total	5·49

It will be thus found that a large proportion of the entries has been recent. While the experience has this disadvantage, it is conceived that its value—more particularly as bearing on the question of the value of selection of lives by Assurance Offices—is, on the other hand, much enhanced by the non-hazardous lives being stated separately—by the non-hazardous male lives being kept separate from the female lives—and by the lives at risk and the deaths being stated both for each year of the assurances and for each year of the Society.

It is unnecessary to allude to the well-known very great labour which is involved preparatory to the exhibition of tables of this nature. It is proper, however, that I should notice—

1st. That the policies were first arranged according to the years of birth of the lives assured, which gave facility for deleting policies in cases where more than one was held on the same life, so that not more than one of these should be reckoned at risk at the same time.

2nd. That the ages at entry, as for the first year's experience of a life, were taken as the difference between the year of birth and the year of entry—a mode which evidently (on the assumption, at least, of parties being born equally at any time of a year) states the ages on an average correctly.

3rd. That in the case of policies cancelled otherwise than by death, when the broken period was less than half a year, 0 was reckoned; when half a year, ·5; when more than a half, 1.

The work was ably executed by, or under the immediate personal superintendence of, the Secretary of the Society, with the assistance of a careful clerk. Besides that I think it due to Mr. Stott to make this statement, I consider that it is desirable that parties who may refer to, or make use of, the experience, should be aware that the details have this voucher for their correctness.

The tables show—

For each year of the Society, and of the Assurances, the lives at risk and the deaths in the above four classes—abstracts, in quinquennial groups of ages, following each table.

For each year of the Society, and of the Assurances, the deaths, actual and expected by the English Life Table No. 2, for non-hazardous males and females separately—and by the Northampton Table, for non-hazardous males and females jointly—in quinquennial groups of ages.

Totals at risk and deaths, actual and expected by various tables of mortality, arranged according to the classes in quinquennial groups of ages.

And there are added—

Nosological Tables, classified, in accordance with 20th and previous Reports of the Registrar-General, by Dr. Fleming, the Medical Officer of the Society, in decennial groups of ages, showing the special causes of death at the different ages; and, as regards the general classes of death, the duration of the assurances at the different ages; also, in the case of the non-hazardous males, proportioning the expected deaths by the English Life Life Table according to the deaths, 1848 to 1854, in Registrar-General's 18th Report—showing the shortcoming or excess of the actual deaths.

It was thought better to present the Tables without comment, and that I should submit to the Institute my observations upon some points which I think these and other Tables illustrate. It may be thought that in some respects the Tables are too minute and too detailed—and, if our own experience were to be solely

looked to, I dare say they might be considered too elaborate—but it is undoubtedly by such subdivisions and details that advances in our knowledge of the subject are to be made; and I feel confident that, as regards after combinations and comparisons which may be made with the like experience of other Offices, the labour will be found to have been well bestowed.

It appears from Table TD, that, during the whole period of the experience, the actual deaths in the Scottish Amicable Society have been the following percentages of calculated deaths by various tables of mortality:—

Table I.

	Percentages which Actual Deaths are of Deaths by various Tables.			
	Northampton.	E. L., No. 2.	Carlisle.	Experience, Table G.
Males—Non-hazardous	53·95	81·34	88·09	91·17
Females—Non-hazardous	54·08	82·63	84·96	85·56
Hazardous—Both sexes (exclusive of West Indies)	96·91	145·44	157·46	162·86
West Indies	95·34	153·61	163·09	177·68
Total	56·48	85·34	91·98	95·06

The above does not show any extra mortality among the female lives. An examination, however, of the details of the experience exhibits the following result:—

Table II.

	Percentage which Actual Deaths are of Deaths by English Life Table, No. 2, Males and Females respectively.	
	First Six Years.	After Six Years.
Males—Non-hazardous	69·7	98·2
Females—Non-hazardous	79·2	89·3

The observations on female life are too limited to draw a conclusion with any confidence, but it is believed that during the currency of the experience a greater proportion of ineligible lives has been proposed among females, and there are probably greater difficulties in the medical examination. Such causes would undoubtedly operate in the way of producing the result above indicated—viz., a greater percentage of deaths among the females at

first, followed, after the worst lives may have died out, by confirmation of the superior vitality of the females.

The experience of the West India risks cannot fail to be interesting to the Institute. It will be seen that the mortality is not far from double that of the non-hazardous males, and is about equal to that according to the Northampton Table. On the one hand, I am willing to believe that the percentage of deaths experienced is beyond what may be expected in favoured spots; and, on the other hand, it must be kept in view that the experience of these risks is only for an average of 4·35 years, and the prevalence of some severe tropical disease may yet materially add to the rate of mortality.

It is to be observed, in reference to the deaths among the hazardous lives, exclusive of the West Indies, that four of them occurred in a small Friendly Society for assurances which the Scottish Amicable took up—viz., two in the first year of the assurances and two in the second—of lives which would not otherwise have been assured.

The total number exposed to risk, of the whole assurances, during the first year, has been 10,109·5; and the total deaths during the first year, including the two noticed above and at least two deaths by suicide, 73—showing a percentage of ·72 or ·68, exclusive of above four deaths. This percentage does not appear to be below what might have been expected in a Society where no large proportion of the risks has been accepted before the period when careful medical examinations were made. It is above the percentage (·55) of sums assured becoming claims in the first year, deduced from an amount of about fifty millions assured. (See *Assurance Magazine*, vol. iv., p. 4.) It is right that I should mention that it is considerably under the experience which has been furnished me of an old Office, the deaths being 90 out of 7,419, and the percentage thus 1·213. From Mr. Neison's valuable analysis of the Gotha experience, in his *Contributions to Vital Statistics*, it would appear, from Tables VIII. and XV., pp. 165 and 191, that the percentage of deaths in the first year, 1840 to 1849, is about ·85. The Manager of an extensive American Assurance Office has been so good as to send me a note of the number of policies which have been exposed to risk during the different years of the assurances and the policies becoming claims; and I think it may be well that I exhibit this, along with similar experience of the Scottish Amicable, for the first six years:—

Table III.

American Office Policies.			Year of Assurances.	Scottish Amicable Lives.		
No. Exposed.	Deaths.	Per Cent.		No. Exposed.	Deaths.	Per Cent.
14,869	102	·68	1	10,109½	73	·72
11,902	112	·94	2	8,393	83	·98
9,873	91	·92	3	7,082½	82	1·15
8,164	102	1·25	4	5,875½	62	1·05
6,780	81	1·19	5	4,728	58	1·22
5,650	60	1·06	6	3,886½	58	1·49

It will be observed that the percentage in the American Society in the first year is ·68; that of the Scottish Amicable, ·72. In the second year the percentages are ·94 in the American Society and ·98 in the Scottish Amicable. In the experience of the London Equitable, involving as it does 21,398 lives, the deaths during, not the first year of assurance, but the first calendar year after entry, seems to be represented as 105—showing, after giving effect to discontinuances, a percentage of ·491. While, I believe, this is very near the percentage on non-hazardous lives as presently selected during the first year of the assurances, and the percentage in such Offices as deal little, or not at all, with hazardous cases, the percentage thus stated for the Equitable for the first calendar year after entry, contrasted with that of the second year, 1·167, and 245 the number of deaths, is very startling. I find that, in the Scottish Amicable, the deaths in the *first calendar year after entry* are 83; and, in the *second calendar year after entry,* 77—being, in the first year, a percentage of about ·9; and, in the second, about 1 per cent. It seems to me to be of very great importance that it should be ascertained if there is any explanation of the extraordinary result thus exhibited by the London Equitable experience. Is it not possible that some anomaly may have been produced in noting down the deaths for the first and second years, connected with the exclusion which is seemingly stated to have been made of the lives at risk during the first calendar year of entry? Mr. Morgan, in a Note on p. iv. of the experience, says:—

"One hundred and eighty-five new members were admitted into the Society in the course of the whole year 1829, at various ages; these are not included in the Table A, *although the mortality of that year is included in that table.* The reason for this omission will be apparent from a consideration of the principle upon which tables of this kind are formed and of the information they are intended to convey—namely, the number of deaths

which have happened, or may be expected to happen, out of a certain number of individuals, at any given age before they shall attain to the next year of age. In conformity with this principle, the duration of the lives of those who are or have been members of the Society, is reckoned according to the number of entire years of age which they have completed since the time of their admission.

"A person, therefore, making assurance in the course of the year 1829, in the 31st year of his age, cannot be reckoned as having lived a year in the Society on the 31st day of December, 1829; and is not to be added until 1830, when he must be classed under the age of 31, as he would complete his 32nd year in 1830, and have entered his 33rd year on the 1st of January, 1831, should his life have so long endured."

The following Note is appended in the article "Mortality, Human," by the late Mr. Milne, in the *Encyclopædia Britannica*, page 639 of vol. xv. of last edition:—

"None of the lives insured during any year commencing with the 1st of January are entered in these statements as having come into the Society until the 1st of January next following the day of their actual admission, and their age is stated to be a year older than it was when insured; *although whatever deaths may happen amongst them before the 1st of January next following the commencement of the insurances, are entered in the statements as having taken place amongst the lives previously insured.*"

Mr. Edmonds, in his paper in the *Lancet*, 1842, says:—

"He [Mr. Morgan] assumes that all who have been admitted during any year of the observation, in any specific year of age, will have completed that year of age at the end of such year of the observation. Having made this assumption, he disregards any mortality occurring in the fractional part of the first year of admission, and commences the observation from the beginning of the ensuing year, when the age is expressible without fractions of a year. For example: if a member had been admitted in the 45th year of age in any part of the year 1821, he will have been regarded as having completed his 45th year, or as being exactly 45 years old, on the 31st of December, 1821; and, for the purposes of the observation, no account will have been taken of this member before the 1st January, 1822. Although, as I have learned from inquiries, this is the mode of proceeding adopted by Mr. Morgan, yet there is no statement to this effect published by him."

The above is the whole which I can discover printed on this subject. Apparently, the mortality for the year 1829, noted separately, page 29 of the Equitable experience, is not included in the mortality in Table A, as it would seem it ought to be, and as it is stated by Mr. Morgan to be; and it would confer a great additional boon if the two points here noticed were carefully looked into again, and an authoritative statement made on the subject.

The observations which have preceded have had more especial reference to the general mortality; I propose now to consider, somewhat more in detail, the mortality among assured male lives non-hazardous—say select assured male lives.

And first, in regard to the mortality in the first year in the Scottish Amicable, the number exposed to risk has been 8,482, and the deaths 47—the percentage thus being ·554. Seeing, as we have stated above, that the percentage of sums assured in the Offices above referred to, including any hazardous risks, is ·55; there seems no reason to doubt that this ·554 is not below the probable general mortality for select assured male lives during the first year. The percentage which should have fallen by the English Life Table, No. 2, is 1·245—the Scottish Amicable (select male) deaths being 44·4 per cent. of this. That the 1·245 per cent., or 1,245 out of 100,000, includes a large proportion of lives not assurable within the year, appears also clearly from the following important observation which, some years ago, I was enabled to make by the kindness of medical gentlemen connected with the Scottish Amicable Society or personal friends of my own. I furnished these gentlemen with schedules as to deaths of parties above the age of 13, and asked them to fill in such deaths as had occurred in their practice, giving the cause, about the age, and stating "Yes" or "No" to the question whether the life was assurable as a good life within a year of death.

I had these returns filled up with 513 cases, of which

200	were stated to have been assurable as good lives within a year of death, and
313	were stated as not assurable as good lives within a year of death.
513	

The following Table IV. shows, according to the Registrar-General's Reports referred to, the number of deaths, in the different classes of diseases among the lives assurable and not assurable as good lives within the year, at ages under 48, from 48 to 62 inclusive, and above 62.

It will be seen how much the proportion of deaths of lives non-assurable within the year, as good lives, increases with each period of age; and the marked feature of the great preponderance of deaths in the zymotic, &c., class (I.) in the assurable division, and of tubercular (III.) diseases in the non-assurable division, at ages under 48, will at once attract attention.

TABLE IV., *showing the Division, according to the Registrar-General's Classification* (20*th and previous Reports*), *of* 513 *Deaths reported by Medical Gentlemen, of which Deaths* 200 *were reported as assurable within a Year of Death, and* 313 *not so assurable.*

Class of Disease.	278 Lives under Age 48.		113 Lives 48 to 62, inclusive.		122 Lives above Age 62.		513 All Ages.	
	Assurable.	Not Assurable.	Assurable.	Not Assurable.	Assurable.	Not Assurable.	Assurable.	Not Assurable.
I.	55	5	15	7	7	8	77	20
II.	6	8	2	14	1	10	9	32
III.	16	60	1	7	1	..	18	67
IV.	14	9	1	11	5	17	20	37
V.	5	11	1	14	1	11	7	36
VI.	19	11	9	10	3	20	31	41
VII.	10	8	4	8	3	9	17	25
VIII.	2	7	..	4	.	9	2	20
IX.	12	4	..	3	..	..	12	7
X.	1	3	..	..	..	..	1	3
XI.	..	..	..	..	..	..	..	..
XII.	..	..	..	..	..	..	..	..
XIII.	..	1	..	..	..	5	..	6
XIV.	2	1	..	..	..	1	2	2
XV.	..	..	..	..	..	9	..	9
XVI.	..	..	..	..	..	..	..	..
XVII.	3	5	..	2	1	1	4	8
Totals..	145	133	33	80	22	100	200	313

I do not at present propose to analyse the details further, nor to argue for any precise accordance of the results with the experience of select lives for the first year, but it is obvious that they distinctly confirm the view that a very large proportion of the deaths among the population, as usually shown in mortality tables, are of lives not assurable as good lives within the year.

In reference to the apportionment of a general rate of mortality for the first year, similar to what is stated above, I have been unable, on a careful consideration—or, I may say, reconsideration—to discover whether, up to the age of about 45, the mortality is higher or lower at younger ages; and, in the meantime, I have stated it, up to age 42, at ·45. I do not say that the recent valuable paper on the mortality of the peerage, *Assurance Magazine*, vol. ix., p. 311, added to the paper on the vital statistics of the Society of Friends, read before the Statistical Society, 21st Dec., 1858, and the mortality exhibited among the Government annuitants, lead undoubtedly to the inference that such a result as the above is more likely to give undue advantages regarding vitality to the younger than the older lives, because it is quite possible that

such results as the peerage mortality shows may be consistent with an increasing mortality among select lives; but I think they may tend to promote opinions that the views here suggested for consideration are, at least, not unworthy of serious and careful attention.

Having made these observations in reference to the rate of mortality for the first year, I now proceed to consider the mortality for after years; and first, in regard to mortality in later years of the assurances. Following a similar process of argument to that adopted by Mr. Edmonds in his papers on the London Equitable Society in the *Lancet*, I believe that after six years it may be assumed that the mortality is in accordance with that of the English Life Table, No. 2; and it will be seen from the following table, that the total deaths in the London Equitable, "Town Males," and Scottish Amicable male lives non-hazardous, after the first six years, agree almost exactly with the English Life Table.

Table V.—*Deaths, Actual and Expected, by English Life Table, No. 2, for first Six Years, and thereafter, in the case of the London Equitable, "Town Males," and the Scottish Amicable Males, non-hazardous.*

	Deaths during first Six Years of Assurance.				Deaths after first Six Years of Assurance.			
	Actual.	Expected.	Excess of Expected Deaths.	Percentage of Excess on Actual.	Actual.	Expected.	Difference of Expected Deaths.	Percentage of Diff. on Actual.
Equitable	1,208	1,546·025	338·025	27·98	3,936	4,067·343	+ 131·343	+ 3·33
Do. if 176, the deaths for 1829, as mentioned above, be added (say 16 within six years and 160 after)	1,224	1,546·025	322·025	26·30	4,096	4,067·343	− 28·657	− ·68
"Town Males"—(doubling, for first year, deaths in first calendar year)	710	901·685	191·685	27·00	480	*467·315	− 12·685	− 2·63
Scottish Amicable Male Lives, non-hazardous	322	461·334	139·334	43·27	310	315·659	+ 5·659	+ 1·82

This is a result in itself accordant with what might be expected; for while, in six years, lives that may have been bad at first should have disappeared, still there will be a tendency always for bad

* Approximated.

lives to be continued rather than good ones; and a few bad lives continued will add to the rate of mortality more than the diminution from some hundreds of good lives discontinued. These causes will tend to assimilate different experiences in after years; and it will be seen from this table how, notwithstanding the superiority of the Scottish Amicable select male lives in the first six years, they come to agree very nearly in after years. It is unnecessary for me to say how desirable it is for parties anxious to study the value of selection to refer to Mr. Higham's valuable papers.

To come to a conclusion I found it necessary to adopt some such assumption as that above stated, in reference to the mortality after the first six years; and while, doubtless, future investigations will lead to more precision, yet I am persuaded that any correction which may be made can but little affect the values of annuities calculated on this assumption.

Having thus so far considered the rate of mortality for the first year, and the rate after six years, I submit the following table, made out with the view of assisting me to the concluding column of percentages, for the first six years, at ages 22, 27, 32, 37, 42, 47, 52 and 57, for select assured male lives—the observations being founded on lives of five ages where these were the medium ages. I have also represented this table by the Diagram No. 1 appended to this paper.

TABLE VI.—*Percentage of Deaths according to the Experience of the Equitable Office* (E.), "*Town Males*" (T. M.), *and the Scottish Amicable Non-hazardous Male Lives* (S. A.), *and Percentage deduced during the first Six Years of the Assurances.*

Age at Entry.	Age.	1. E.	2. T. M.	3. S. A.	Average of 1, 2, 3.	Percentage Deduced.
22	22	·288	·173	·457	·306	·450
	23	1·174	1·247	·943	1·121	·600
	24	·718	1·222	·953	·964	·725
	25	1·336	·673	·297	·769	·800
	26	·726	·206	·762	·564	·850
	27	·818	·755	1·478	1·017	·895
		5·060	4·276	4·890	4·741	4·320
27	27	·348	·480	·447	·425	·45
	28	·720	·505	·705	·643	·65
	29	·666	·937	1·641	1·081	·80
	30	·747	1·109	·358	·738	·88
	31	·765	·454	1·356	·858	·94
	32	·917	·862	1·278	1·019	·99
		4·163	4·347	5·785	4·764	4·71

Table VI. (continued).

Age at Entry.	Age.	1. E.	2. T. M.	3. S. A.	Average of 1, 2, 3.	Percentage Deduced.
32	32	·185	·540	·336	·354	·45
	33	·754	·724	·605	·694	·70
	34	·840	·993	·548	·793	·86
	35	·848	·746	·746	·780	·96
	36	·990	·928	·915	·944	1·04
	37	·952	1·351	·840	1·048	1·10
		4·569	5·282	3·990	4·613	5·11
37	37	·453	·797	·756	·668	·45
	38	1·300	·998	·528	·942	·73
	39	1·589	1·151	·176	·972	·90
	40	1·158	·536	1·244	·979	1·05
	41	1·170	1·024	1·167	1·120	1·18
	42	1·254	·851	1·896	1·334	1·30
		6·924	5·357	5·767	6·015	5·61
42	42	·366	·364	·423	·384	·45
	43	1·086	1·138	·996	1·073	·76
	44	1·048	·467	·919	·811	·96
	45	1·131	1·570	1·492	1·398	1·15
	46	1·163	·835	1·138	1·045	1·33
	47	1·186	2·112	1·165	1·487	1·50
		5·980	6·486	6·133	6·198	·615
47	47	·765	·658	·824	·749	·67
	48	1·428	·827	1·150	1·135	1·
	49	1·610	1·249	1·918	1·592	1·24
	50	1·917	1·270	2·005	1·731	1·46
	51	1·762	1·969	·278	1·336	1·66
	52	1·890	3·136	1·278	2·102	1·85
		9·372	9·109	7·453	8·645	7·88
52	52	1·343	·627	1·389	1·120	·80
	53	1·615	1·552	1·070	1·412	1·15
	54	2·230	3·811	1·235	2·425	1·49
	55	1·980	2·221	·728	1·643	1·82
	56	1·936	1·827	2·511	2·091	2·14
	57	2·185	3·968	1·990	2·714	2·45
		11·289	14·006	8·923	11·405	9·85
57	57	1·037	1·721	·791	1·183	·90
	58	2·244	2·505	1·380	2·043	1·44
	59	2·174	1·657	2·117	1·983	1·97
	60	2·220	4·514	3·077	3·270	2·49
	61	2·830	1·690	2·256	2·259	3·00
	62	3·379	2·372	1·770	2·507	3·50
		13·884	14·459	11·391	13·245	13·30

In reference to the Equitable experience, I have indicated that I have doubts of the first year's mortality; and as regards the

"town males," the first year's mortality being represented by double the deaths during the first calendar year, it may be thus understated; but both these experiences include, I understand, any hazardous lives which may have been accepted—though probably these will be few—and they are the experience of a considerable period during which medical examination was not made at all, or in a much less strict mode than more recently; and, indeed, while the Equitable experience for the first year shows a percentage of only 38·1 of the calculated deaths by the English Life Table, and the Scottish Amicable 44·4 per cent.; in the remaining five years, on the other hand, the Equitable shows a percentage of 86·9, and the Scottish Amicable 77·3. The "town males" again, by doubling the deaths as above stated, show a percentage of 53·9 during the first year, and 85·8 during the next five years, all as shown more in detail in Table VII., which has been made out at considerable trouble, and will, it is thought, be worthy of a careful perusal. (See page 74.)

Before passing from Table VII., I may notice that it pretty clearly indicates that the percentage on assured lives, according to the English Life Table, No. 2, will be for 10,000—

For 1st year . . .	from 125 to 130
For 2nd year . . .	,, 130 to 135
For 3rd year . . .	,, 135 to 140

Returning to Table VI., I may state, that while, in the results deduced in it, I have desired rather to follow the Scottish Amicable, it will be seen that they are very much in accordance with the average of the three tables; and I think it will be found difficult, on a careful consideration of the whole table, to believe that there could be a material alteration on the *sums* of the percentages deduced for the six years; and, if so, whatever doubts may be entertained of the divisions of the totals, it is certain that no alteration which could be made would, at all events, affect to any considerable extent the values of the annuities to be deduced. In stating the percentages for the first six years, I should add that I have rather leaned to the side of what may be called safety; and, more particularly as regards the advanced ages, and 57 in particular, I think the percentages might have been less, and probably the falling into the English Life decrement retarded; but without more clear grounds, I thought it better, for simplicity, to adhere to the merging in the English Life Table at all ages after six years.

In regard to the values of annuities for select assured male lives, I think it may be useful, before submitting a new table of

Table VII., *showing, in reference to London Equitable Experience, "Town Males," and Scottish Amicable Males—non-hazardous, the Deaths, Actual and Expected, by English Life Table, No. 2, during first Six Years and thereafter; also the Percentages which the Deaths, Actual and Expected, bear to the Lives exposed to Risk.*

Years of Assurance.	London Equitable.						Town Males.						Scottish Amicable.					
	Expected Deaths by E. L., No. 2.	Percentage on Lives at Risk.	Lives at Risk.	Actual Deaths.	Percentage on Lives at Risk.	Percentage of Expected Deaths, E. L., No. 2.	Expected Deaths, E. L., No. 2.	Percentage on Lives at Risk.	Lives at Risk.	Actual Deaths.	Percentage on Lives at Risk.	Percentage of Expected Deaths, E. L., No. 2.	Expected Deaths, E. L., No. 2.	Percentage on Lives at Risk.	Lives at Risk.	Actual Deaths.	Percentage on Lives at Risk.	Percentage of Expected Deaths, E. L., No. 2.
1	275·238	1·287	21,382	105	·491	38·148	200·264	1·299	15,406	*108	·701	53·929	105·641	1·245	8,482	47	·554	44·490
2	278·680	1·344	20,739½	245	1·181	87·914	181·616	1·353	13,424½	142	1·058	78·185	91·998	1·300	7,079	61	·862	66·305
3	266·971	1·408	18,961½	239	1·260	89·523	155·883	1·400	11,135	145	1·302	93·020	80·979	1·352	5,991	63	1·051	77·797
4	250·911	1·465	17,129½	216	1·261	86·086	134·669	1·406	9,361	117	1·250	86·879	70·614	1·409	5,013	52	1·037	73·639
5	241·246	1·532	15,748½	200	1·270	82·902	120·495	1·503	8,015½	87	1·085	72·205	59·956	1·473	4,069½	50	1·229	83·393
6	232·979	1·598	14,578½	203	1·392	87·132	108·758	1·566	6,946	111	1·598	102·050	52·146	1·550	3,365	49	1·456	93·965
First 6 years	1546·025	1·424	108,539½	1,208	1·113	78·135	901·685	1·400	64,388	710	1·102	78·742	461·334	1·357	33,999½	322	·947	69·798
Five years after 1st	1270·787	1·458	87,157½	1,103	1·265	86·796	701·421	1·432	48,982	602	1·229	85·827	355·693	1·394	25,517½	275	1·078	77·314
Total after six	4067·343	2·647	153,670½	3,936 † or 4,096	2·561 or 2·665	96·771 or 100·70	467·315	. .	. .	480	. .	102·71	315·659	·928	14,348½	310	·912	98·207

* Double the deaths in first calendar year.

† In this table the deaths for the first six years are stated without reference to the 16 deaths noticed in Table V.

these, to illustrate, by a diagram (see Diagram No. 2), values at 4 per cent., which I deduced, in 1844, from the Equitable tables. I think that the table then formed must be, as near as possible, the true value of annuities for select lives, upon the assumption that these may be deduced from the Equitable experience as published. One table of the values of annuities was calculated according to the actual probabilities of living by Morgan's Table A. They were also calculated for the separate lives from 25 to 35, 35 to 45, 45 to 55, and 55 to 65—the calculations being made by the probabilities of living one year given for these lives in the Tables of the Experience of Seventeen Offices. It was clear that in each of these separate tables the value at the commencing age should be too great and those at the end too small. It was also clear that the real values at 25, 35, and 45, should be between the values stated in the different separate tables. It was further explained, in reference to the table alluded to, how the difference in the value of the annuities for select lives at one age and the next should be less than the differences between one age and the next in the Table of Annuities calculated from the probabilities by Morgan's Table. The Table of Annuities was made out having all these points in view—the principal difficulty being in reference to the values from 39 to 45, which it was found necessary to state nearly in accordance with the separate lives entered between 35 and 45. I have shown in the diagram these values by their differences from the values deduced in 1844, also the values as calculated by Mr. Davies, and also by Mr. Higham, at intervals of age, the values at 4 per cent. being kindly furnished to me by the latter gentleman by approximation from the values he deduced at 3 per cent. (See Mr. Higham's paper, read before the Institute 25th March, 1850.)

I have also shown in this diagram the values now deduced (Tables VIII. and XI.), and at intervals, values of Mr. Higham approximated at 4 per cent. from the values given in the second most able paper which he read on selection. (See *Assurance Magazine*, vol. i., p. 179.)

The values of annuities have again been calculated (see Table VIII.) for ages 22, 27, &c., up to 57, on the assumption of the mortality in the London Equitable, after six years, being in accordance with the English Life Table, and taking the actual mortality prior to this. The differences of the two values, as seen in this table, are not very great—being greatest at ages 37 and 47, where they are ·28 and ·21; and I think it may be held, so far as regards the Equitable Society, that no material error in calculating

the values of annuities at separate ages can arise by proceeding on the general assumption of agreement with the English Life Table after six years. It will be seen that the last calculated values are generally the lower, but if the deaths of 1829 be not included in Table A, as previously alluded to, page 67, the values of 1844 should have been stated less. The table shows, at ages 22, 27, &c., to 57, the values of the annuities in 1844 as calculated from Equitable experience, the values deduced of new from Equitable experience as above stated, the values of "town males" and Scottish Amicable on the same principle, and the values of annuities according to the rate of mortality deduced for the first six years and followed by that of the English Life. These last values will be found to differ in no material degree from those deduced from the Equitable in 1844, and, as seen in Diagram No. 2 and Table XI. below, they are almost identical with Mr. Higham's of 1851, deduced by a considerably different process; and, upon the whole, I think that it will be evident that they cannot be far from the truth—that is, they must be very near the real values of annuities for assured male lives non-hazardous. The principal feature of the values of annuities in 1844 was the high value it gave to advanced lives towards 60, and the comparatively modified value it assigned to young lives. The further investigations made do not contradict this. I need not add, that one great advantage of a true table of such annuities is the measure it enables us to take of the fairness of modes of allocations of additions.

TABLE VIII.—*Values of Annuities at 4 per Cent. on Observation of Select Assured Lives.*

Ages.	Values deduced from Equitable Experience, 1844.	Values calculated by Actual Mortality for Six Years, and after by English Life Table, No. 2.			Values per Mortality deduced in Table II. for first six years, and after by E. L., No. 2.
		Equitable.	"Town Males."	Scottish Amicable Males, Non-hazardous.	
22	17·90	17·81	17·92	17·84	17·93
27	17·29	17·32	17·29	17·08	17·24
32	16·55	16·52	16·44	16·63	16·47
37	15·69	15·41	15·60	15·59	15·60
42	14·65	14·64	14·59	14·67	14·61
47	13·45	13·24	13·31	13·42	13·41
52	12·12	11·94	11·71	12·17	12·10
57	10·74	10·65	10·55	10·85	10·72

If what we have given above be the values of select assured lives, there is no doubt, in reference to physiological inquiry and the values of annuities on select lives continued under observation during the whole period of these lives, the values will be repre-

sented too low and the percentages of mortality too high. In the first place, there will certainly be some bad lives accepted by Assurance Offices as good lives. The great damage from this may probably come forth principally in the first year; but of the 47 deaths which took place among the Scottish Amicable male lives, "non-hazardous," there were, undoubtedly, at least 8, and perhaps 12, more or less, bad—2 or 4 of these being suicides in the first year of the assurance, out of 7 or 9, the whole number of suicides in the Society's experience. It would seem that, taking hazardous and non-hazardous together, the deaths to be expected for the first year may be from 6 to 7 per 1,000; and, for select assured lives, about 5 per 1,000, one of which would likely be a life really ineligible as a good one. In the second place, there is, as noticed, the tendency of the bad lives to be less discontinued than the good ones. I have little doubt that, if the materials at present available were made use of, both values could, with very little difficulty compared with the importance of the subject, be ascertained very nearly; and I cannot think the value of annuities on select assured male lives will be found to differ in any considerable degree from those stated here. As regards the values of annuities on select lives continued under observation during the whole of the lives, it is very satisfactory to be able to refer to that most valuable addition made by Dr. Farr to his contributions to the science in the Life Table for Healthy Districts. Mr. William Davis, in the last number of the *Assurance Magazine,* has made out a D and N Table from this at 3 per cent., by which I have been enabled to contrast the values of the annuities at 3 per cent. with values at the same rate which I have deduced as for select assured male lives. The following are the results:—

TABLE IX.—*Values of Annuities at 3 per Cent.*

Ages.	Select Assured Male Lives.	Healthy Districts. Farr.
22	21·047	21·909
27	20·057	21·032
32	18·975	20·039
37	17·786	18·856
42	16·484	17·553
47	14·964	16·029
52	13·342	14·311
57	11·688	12·389

It is thus seen that the values of annuities by Dr. Farr's "Healthy Districts" Table considerably exceed the values for select assured male lives. There is every reason to suppose that lives

selected in these districts, and continued under observation for the whole period of life, would give still higher values; but I doubt if any practical selection of lives which has been made will be found to give higher values. I now subjoin—

Table X.—*Rates per Cent. of Mortality for Select Assured Male Lives, for the first Six Years of the Assurances—the Rate thereafter being assumed to be according to the English Life Table, No. 2.*

Age.	1st Year.	2nd.	3rd.	4th.	5th.	6th.	
22	·45	·60	·725	·80	·85	·895	
23	·45	·62	·74	·82	·85	·91	
24	·45	·63	·75	·84	·89	·93	
25	·45	·64	·78	·85	·91	·96	
26	·45	·65	·79	·87	·92	·98	
27	·45	·65	·80	·88	·94	·99	
28	·45	·66	·81	·90	·97	1·01	
29	·45	·66	·82	·93	·98	1·03	
30	·45	·68	·84	·93	1·00	1·06	
31	·45	·69	·84	·94	1·02	1·08	
32	·45	·70	·86	·96	1·04	1·10	
33	·45	·70	·86	·98	1·08	1·14	
34	·45	·70	·87	1·00	1·10	1·18	
35	·45	·70	·88	1·01	1·13	1·22	
36	·45	·72	·90	1·03	1·16	1·26	
37	·45	·73	·90	1·05	1·18	1·30	
38	·45	·74	·91	1·08	1·21	1·34	
39	·45	·75	·93	1·10	1·24	1·38	Thereafter English Life Table, No. 2.
40	·45	·75	·94	1·10	1·28	1·42	
41	·45	·76	·95	1·13	1·30	1·46	
42	·45	·76	·96	1·15	1·33	1·50	
43	·49	·81	1·01	1·21	1·39	1·58	
44	·53	·86	1·06	1·26	1·46	1·64	
45	·58	·91	1·12	1·34	1·52	1·70	
46	·62	·96	1·18	1·39	1·59	1·78	
47	·67	1·00	1·24	1·46	1·66	1·85	
48	·70	1·02	1·29	1·52	1·76	1·96	
49	·73	1·06	1·34	1·60	1·86	2·04	
50	·76	1·05	1·39	1·68	1·95	2·15	
51	·78	1·12	1·41	1·75	2·04	2·28	
52	·80	1·15	1·49	1·82	2·14	2·45	
53	·82	1·21	1·58	1·95	2·32	2·66	
54	·84	1·26	1·66	2·08	2·49	2·88	
55	·85	1·33	1·78	2·22	2·68	3·08	
56	·88	1·38	1·86	2·36	2·84	3·28	
57	·90	1·44	1·97	2·49	3·00	3·50	

This contains the adjusted column in Table VI., and the intermediate rates are interpolated.

The following Table XI. is calculated at the ages for which the rate of mortality is adjusted in Table VI. by those rates of mortality for the first six years, and English Life Table thereafter. The values for the intermediate ages are interpolated, but I have no doubt these would be very little different if the annuities had been calculated in detail from the preceding table; and it will be seen that the values fall in accordance with what has been

said to be stated as almost the same as those which I made out in 1844. They are both represented in Diagram No. 2, as above mentioned; and to impress their agreement also with Mr. Higham's, of 1851, I add here his approximate values, as shown in the diagram.

TABLE XI.—*Values of Annuities for Select Assured Male Lives at 4 per Cent.*—1861.

Ages.	Value of Annuity.	Ages.	Value of Annuity.	Ages.	Value of Annuity.	Ages.	Higham, 1851.
20	18·16	34	16·14	48	13·16	25	17·42
21	18·04	35	15·96	49	12·90	30	16·75
22	17·93	36	15·78	50	12·64	35	15·95
23	17·80	37	15·60	51	12·37	40	15·02
24	17·67	38	15·41	52	12·10	45	13·87
25	17·53	39	15·22	53	11·83	50	12·60
26	17·38	40	15·03	54	11·56	55	11·30
27	17·24	41	14·83	55	11·28	60	9·83
28	17·09	42	14·62	56	11·00		
29	16·95	43	14·39	57	10·72		
30	16·79	44	14·16	58	10·44		
31	16·64	45	13·92	59	10·16		
32	16·47	46	13·67	60	9·89		
33	16·31	47	13·42				

NOTE.—It will be seen that the values of annuities have been given for ages 20 and 21, and 58, 59, and 60. This has been done by observation of the differences in Diagram No. 2. The values now given by me are intended to exclude, more entirely than Mr. Higham, hazardous lives; and his values being generally a little lower, is rather in accordance with this than otherwise.

I may notice, in reference to the 632 deaths which took place among the Scottish Amicable males, non-hazardous, that, by referring to Table RP of the Scottish Amicable experience, it will be seen that, as compared with the deaths to be "expected" by the English Life Table No. 2, the difference 144·99 is thus accounted for.

TABLE XII.—*Shortcoming or Excess of Deaths in the following Classes of Diseases.*

	Shortcoming.		
II.	Diseases of uncertain seat	20·95	
III.	Tubercular disease	89·24	
VI.	Diseases of the respiratory organs	35·84	
XIV.	Atrophy	10·35	
XV.	Gradual decay and senile debility	7·45	
XVI.	Cause of death not ascertained	5·66	
XVII.	Accidental injuries, suicides, &c.	17·39	
	Other classes	13·14	
			200·02
	Excess in the following classes of disease.		
IV.	Disease of the brain and nerves	25·80	
V.	Disease of the heart and blood-vessels	10·02	
VII.	Disease of the digestive organs	17·24	
IX.	Disease of integumentary tissues	1·97	
			55·03
	Difference, as stated	..	144·99

TABLE XIII.—*Approximate Calculation of the Deaths of the Scottish Widows' Fund, Expected, per E. L. No. 2, for Seven Years, to 31st December*, 1859, *and Apportionment of them (as if Males) according to the Deaths in Registrar-General's 18th Report*, 1848 *to* 1854; *and also the Shortcoming or Excess of the Actual Deaths from the different Diseases.*

Ages at commencement	13 to 30.		31 to 40.		41 to 50.		51 to 60.		61 to 70.		71 to 80.		81 to 90.		All Ages.		Shortcoming of Actual Deaths.	Excess of Actual Deaths.
Say Ages at Death	25 to 35.		35 to 45.		45 to 55.		55 to 65.		65 to 75.		75 to 85.		85 to 95.					
CLASS.	Expected.	Actual.	Expected.	Actual.	Expected.	Actual.	Expected.	Actual.	Expected.	Actual.	Expected.	Actual.	Expected.	Actual.	Expected.	Actual.		
I.	14·85	10	36·99	26	43·59	18	34·77	27	23·39	17	7·43	9	·37	0	161·39	107	54·39	..
II.	2·47	0	10·72	6	21·24	8	26·37	21	23·19	13	7·72	1	·36	1	92·07	50	42·07	..
III.	33·72	15	70·76	26	63·08	15	32·36	9	9·67	2	·82	0	·02	0	210·43	67	143·43	..
IV.	5·65	8	22·48	32	31·89	61	35·45	51	31·88	54	10·74	28	·44	0	138·53	234	..	95·47
V.	2·93	2	12·06	6	19·98	27	21·60	51	15·45	33	3·43	16	·08	0	75·53	135	..	59·47
VI.	5·75	2	24·02	20	44·20	25	51·70	36	39·82	32	11·60	15	·51	0	177·60	130	47·60	..
VII.	3·85	6	15·63	21	26·73	29	26·59	33	15·88	13	3·28	13	·11	0	92·07	115	..	22·93
VIII.	1·29	3	4·74	5	6·99	12	7·95	11	8·25	16	2·83	5	·11	0	32·16	52	..	19·84
IX.	·01	1	·03	0	·04	0	·04	0	·03	1	·01	0	·00	0	·16	2	..	1·84
X.	·79	1	2·27	5	3·07	3	2·76	2	1·69	1	·36	2	·00	0	10·94	14	..	3·06
XI.	·08	0	·50	0	·93	0	1·05	0	·72	0	·21	0	·00	0	3·49	0	3·49	..
XII.	·01	0	·01	0	·02	0	·01	0	·00	0	·00	0	·00	0	·05	0	·05	..
XIII.	·16	0	·75	0	1·90	0	2·11	0	·22	0	·00	0	·00	0	5·14	0	5·14	..
XIV.	·14	2	·59	0	1·34	0	8·87	0	11·26	1	·21	0	·01	0	22·42	3	19·42	..
XV.	·00	0	·00	0	·00	0	·01	2	26·96	8	44·00	14	5·22	5	76·19	29	47·19	..
XVI. and not specified	1·36	0	4·89	2	8·43	1	9·87	3	8·10	1	2·61	1	·14	0	35·40	8	27·40	..
XVII.	7·97	4	20·44	10	22·37	8	13·88	6	6·09	1	1·47	0	·08	0	72·30	29	43·30	..
Totals..	81·03	54	226·88	159	295·80	207	275·39	252	222·60	193	96·72	104	7·45	6	1205·87	975	433·48	202·61

NOTE.—It is to be observed, that the actual deaths in this table occurred only in seven years recent experience, and include the deaths among females as well as males, and hazardous as well as non-hazardous lives.

Referring to the statement appended to Dr. Begbie's analysis of the deaths in the Scottish Widows' Fund Office for the seven years ending 31st December, 1859, I have made an approximate calculation of the number of deaths by the English Life Table, No. 2, and make them— . . . . . 1,205·87

The actual deaths being . . .	975·
Difference . . .	230·87

The materials for precisely proportioning these deaths, according to the deaths 1848 to 1854 in the Registrar-General's Report, are not complete, but the preceding table, I think, must be pretty nearly correct; and it will be observed that the excess of deaths from diseases of the brain and nerves (IV.), and heart and blood-vessels (V.), and digestive organs (VII.), also appears in the case of this Society for the seven years referred to, and it is much more strongly marked in regard to the two former classes. These two tables suggest important questions for consideration, but I shall not enter upon them at present—more especially as Dr. Fleming may, probably, allude to them in some remarks on the subject of the causes of death, which, I hope, will be published by him shortly.*

I may, before concluding, refer to Abstract C, p. 160 of Mr. Neison's *Contributions to Vital Statistics.* He shows that, as a general rule, in the case of the Gotha Society (to which and its manager the profession are so much indebted), the period which elapsed from the date of the policies to date of death is less, on an average, in the case of younger than in the case of older entrants; and the same rule appears to prevail in the case of the Scottish Amicable Society; at least, the following is the average duration of policies in the case of deaths, among the male lives, non-hazardous:—

Under 30	5·5 years.
30 and under 40	7·0 ,,
40 and under 50	8·3 ,,
50 and under 60	9·2 ,,

But, in reference to the Scottish Amicable, this, no doubt, principally arises from there being an increasing proportion of lives at risk in the later years of the assurances in the case of the older than in the case of the younger lives. Thus, of the lives

* I may so far anticipate Dr. Fleming's remarks, as to mention that, to a great extent, he attributes the excess in Class VII. to the probable insertion in the Registrar-General's Reports, under Class II., of cases of *dropsy*, which, in the returns of Assurance Offices, would be entered among diseases of *liver*—the producing cause.

under 30, as contrasted with those at from 50 to 60 (when the deaths happen to amount to 114 among both sets of lives), the older lives are, in the

1st year of assurance,		29 per cent. of the younger.	
5th	,,	37	,,
10th	,,	43	,,
20th	,,	53	,,

In concluding my remarks, I trust that, while I have not hesitated to express myself decidedly on some points to which I have given much attention and which appear to me clear, I shall not be supposed not to have felt how much more ably the valuable materials I have had to deal with might have been handled by others. They might have thrown light on points which some may suppose my remarks have rendered more obscure than they have hitherto appeared; but still, if the light has been deceiving, it is better that it should be darkened; and whether it be by pointing to the right or from the wrong direction, we must be so far satisfied when we are able to raise some landmarks, however rude, on the road to truth.

Heppel's Logarithms.

It is now, we believe, nearly four years ago since Professor De Morgan made known to some of the members of the Institute that a Mr. Heppel had constructed some tables of logarithms to a large number of places, and that the manuscripts and some stereotyped plates were about to be sold by his executors. Through the intervention of the Professor, the tables and plates were submitted to the inspection of the gentlemen referred to, and were ultimately purchased by them—a small subscription being made for the purpose—and eventually they were placed in the hands of the writer of this notice, with a view to the publication of them. As some introduction was necessary to such a work, it was his intention to have prefaced the tables with a brief but general discussion of the subject, and to have given some account of the principal tables of the kind now in existence. This, however, he has found it impracticable hitherto to effect, and is now the less disposed to attempt, seeing that so good and so modern an account of these tables has just been published in Mr. Charles Knight's *English Cyclopædia**. At all events, it is likely to be very long before even

* See vol. vii., under article "Table."

the materials already collected could be used by him; and as some curiosity naturally exists as to the extent of the tables constructed by Mr. Heppel, he has thought it desirable to state in a few words of what they consist, and to append a short account of the author himself, written by a lady nearly related to him, and, as it would seem, at Professor De Morgan's request.

The principal table, then, consists of the logarithms to twelve places of the numbers 100,000 to 118,600, and it is this table which has been stereotyped in quarto. The next consists of the logarithms to the like number of places of the numbers 131,000 to 136,636, and the last comprehends the logarithms of the numbers 200,000 to 210,000; the logarithms in this being to fifteen places as far as the number 202,024, and to twelve only thereafter. The tables not stereotyped are in manuscript merely. They are comprised in two volumes, which they nearly fill, and are executed with sufficient neatness; but there is little to show how far dependence is to be placed on the accuracy of any of them. Mr. Heppel, it seems, was, at one time, the actuary of an Office called the "Standard of England," which had but a brief existence before it merged into the "Britannia." The account given of him is as follows:—

"George Hastings Heppel, who calculated these tables of logarithms—more for the occupation of his leisure time, and the pure love of science, than from any idea of a profitable undertaking—was born in the year 1793, and evinced, at a very early age, a decided mathematical talent. When at school, whilst other boys were enjoying the pastimes of the playground, he would be found engaged quietly at his desk with his friends Euclid and Hutton. Being the only son of a man of independent property, he was not bred to any profession; but, at the age of 22, his father established him in a paper-mill, in which, unfortunately, he was not successful. It was given up, and, after seven or eight years, he obtained an appointment under the Commissioners for the adjustment of the Spanish claims, for which employment he was far better fitted than for the ordinary details of business. During that period he gave Mr. Michael Quin, the chief Commissioner, valuable assistance in his work upon "Banking," which he acknowledges very handsomely in his preface to that work. Subsequently to the termination of the Commission, he followed the pursuits of an accountant, and was for some time occupied as an actuary of a Life Assurance Company. He was well known to most of the influential men of high standing in the city of London, and esteemed and appreciated by them for his rare talents and integrity; but, unhappily, in the spring of 1844, the seeds of consumption took so fatal a root, and worked their way so insidiously, that, till within three months of his decease, neither himself nor his family were aware of his dangerous state. To his own great satisfaction, however, he had completed such a portion of his table of logarithms as he deemed sufficient for publication, and made terms with Messrs.

Clowes and Sons for printing and stereotyping them; but, before half the plates were composed, his disease made suddenly such rapid progress as to terminate his existence on the 4th February, 1845."

The following memorandum also appears in the volume containing the figures which have been stereotyped :—

"These logarithms were constructed by my father on the basis of Hutton's logarithms of certain numbers to twenty places. He has not left any account of the method he adopted, but the following particulars appear to be clearly deducible from the books containing the calculations.

"Such as could be directly deduced from Hutton's were calculated from them, and those intervening were filled in from the method of differences. Two orders of differences were taken, and these, as well as the logarithms, were calculated to the thirteenth place of decimals—that is, to one more place than was intended for publication. The third order of differences for 100,000 commences with the figure 8 in the sixteenth place; and thus, in the case where they are largest, those in the second order would only alter by one in the course of 120 numbers. As many as possible, in the earlier part, were calculated by their factors, independently of those adjacent to them, so as to check the accuracy of the work; and afterwards, by division by 9 or 8, or by multiplication by 11 or 12, the later logarithms might be found from the earlier, and thus both would mutually check one another."

This memorandum is signed "G. Heppel," and has the date of Carlisle, Nov. 17th, 1855.—Ed. *A. M.*

On the Construction and Use of Commutation Tables for Calculating the Values of Benefits depending on Life Contingencies. By Peter Gray, *Esq.*, F.R.A.S.*

It is to a Mr. George Barrett, of whom nothing besides is publicly known,† that we are indebted for the principle of the Commutation Tables, and for the method of computing, by means of them, the values of benefits depending on the contingencies of human life. The method was first introduced to public notice, after it had been refused a place in the *Transactions of the Royal Society*, by Mr. Baily, in an Appendix to the second edition of his *Doctrine of Life Annuities*, published in 1813. Mr. Griffith Davies, in a work on life contingencies, published in 1825, by certain additions to the tables, and alterations in their structure, according to Professor De Morgan, "increased the utility and extended the power of the method to an extent of which the inventor had not the least idea." Mr. Barrett's method was also

* Extracted from the *Mechanics' Magazine*, for 1842.

† For some account of Mr. Barrett, see vol. iv., p. 185, of this *Journal*.—Ed. *A. M.*

briefly noticed in the Appendix to Mr. Babbage's Treatise on Life Assurance. The method, as improved by Mr. Davies, has since been treated, and a very large collection of tables adapted to it, for both one and two lives, has been given, by Mr. Jones, in his work on annuities, in the *Library of Useful Knowledge*. But by far the most valuable papers on the subject are two in the *Companion to the Almanack*, for 1840 and 1842, by Professor De Morgan, which contain the materials of many thousand formulæ, applicable to almost every case that can occur. There is also some notice of the method in the article "Reversions," in the *Penny Cyclopædia*, which article likewise is the production, we believe, of Professor De Morgan.

The above-named are understood to be the only works in which the new method is, in any sort, treated. They certainly are not numerous; but they are sufficiently so, and well enough known, to have induced a general adoption of this method, to the exclusion of that previously in use, but for two reasons. The first is, the want of tables adapted to this method; and the second, the want of an *elementary* and *systematic* treatise on the subject. The first of these wants is now amply supplied by the valuable collection of tables published by Mr. Jones, above referred to; but the second still exists. It is no disparagement to the able authors of the works above named, to say that this is the case. Generally speaking, it has not been their object to furnish a treatise of this kind; and they have accordingly taken for granted, on the part of their readers, the possession of a degree of acquaintance with the subject, which very many, to whom the power of using the Commutation Tables would be of the greatest service, certainly do not possess. Professor De Morgan, indeed, expressly says, that all that will be found of demonstration in his articles "is intended for those who are familiar with the subject." Now, although the formulæ, according to this method, are extremely simple, and easily intelligible to any one who is acquainted with the merest rudiments of algebra, and who will take the small degree of trouble necessary to enable him to comprehend the notation employed; yet it is a result of our own experience, which we have no doubt can be amply confirmed by the observations of others, that most people view with mistrust and will not willingly have recourse to formulæ, the *principles* of which they do not understand. And while Professor De Morgan's papers, in particular, have, doubtless, well served the end which, from the remark quoted above, the distinguished author appears to have had principally in

view, we do not see that this forms any good reason why others, to whom a knowledge of this method of computation would be of service, should be prevented from availing themselves of the vast fund of information, regarding this method, which those papers contain.

It is, therefore, as an humble contribution towards the supplying of the deficiency which has been shown to exist, that the present papers are intended; and if they shall serve to render more available than heretofore, to any of the numerous readers of the *Journal*, the valuable papers to which reference has so often been made, the writer will consider himself amply rewarded for his labour. While he thinks that the task of rendering more generally intelligible the principles on which the new method is grounded will not be a difficult one, he trusts that, should his success not be commensurate with his wishes, he may at least be found to have aided in clearing the way for some one better qualified to do justice to a subject which is daily growing in interest and importance.

The peculiarities of the old and the new methods may be here briefly stated. In the old method we are presented with a table of the values of annuities at all ages, which of themselves are rarely wanted, but from which, by operations more or less complex, the values of benefits of all other kinds may be computed. In the new method, on the other hand, we are presented with a table which, by mere inspection, tells us nothing; but from which, while the values of the ordinary benefits can be found by a simple division, those of benefits of the most complex description are found by operations consisting usually of nothing more than one or two subtractions and one division. In point of simplicity, moreover, in the deduction of the various formulæ, the methods admit of no comparison. For the establishment of what, according to the old method, required chapters, a few pages will suffice according to the new.

The opinion we have expressed as to the superiority of the new method, however, will probably be regarded as of little value; and that of Professor De Morgan, which is most unequivocally given, may, perhaps, be objected to, as the testimony of an interested witness, seeing he has bestowed such pains on the elucidation of this method. But Mr. Milne's testimony will certainly not be objected to on any such ground. He is the author of the best treatise that has appeared, or is now likely to appear, on the old method; and therefore his prejudices, if he has any, must be sup-

posed to be all on the side of that method which he has done so much to illustrate. Speaking of Mr. Baily's work, he says (*Encyclop. Brit.*, seventh edition, article "Annuities," vol. iii., p. 200) :—"In an Appendix to it . . . . formulæ were given for calculating from tables of that kind [Commutation Tables] the values of temporary and deferred life annuities and assurances, when the annuity, instead of remaining always the same, increases or decreases from year to year by equal differences, *with considerably greater facility and expedition than the same things could have been done with by the tables and methods of calculation in previous use.*" And his testimony, be it observed, refers to the tables as devised by Mr. Barrett, and gives but a faint idea of their capabilities in their improved form.

The construction of the Commutation Tables is effected by combining in a particular manner (which will be explained hereafter) the rates of mortality and interest; and, as in the tables adapted to the old method, any rates that are most approved of, as regards these elements, may be employed. But whatever may be the rates of interest and mortality made use of, the demonstrations and formulæ which will be hereafter given, being generalized by the employment of symbols, will be equally applicable to all tables of the same form.

The rate of interest according to which our table (which has not been heretofore published) is constructed is 4 per cent.; and the rate of mortality is that given by Mr. Finlaison in his "Twentieth Observation on the Mortality of the Government Annuitants" (Parliamentary Paper, No. 122, 1829, p. 58). Tables of mortality, in their usual form, exhibit the numbers who, out of a number supposed to be alive at birth, or some other early age, attain each successive year of age, and consequently also the numbers who die in each year; and this form is the most convenient for the construction of Commutation Tables. But this is not the form in which Mr. Finlaison's data are arranged. What he gives are the *probabilities*, or rather the *logarithms of the probabilities*, that a life at each age will survive a year. A preliminary step, therefore, was, by means of these probabilities, to construct a mortality table of the more usual form; and in doing so, as well as in the subsequent construction of our Commutation Table, Mr. Finlaison's data have been made use of to *their full extent.* It is thought proper to mention this, because there is an "*abridged*" mortality table of the usual form, deduced from Mr. Finlaison's data, published in the *Report of the Select Committee on Friendly Societies,*

1827, p. 82; and some might be embarrassed by finding that the results of our Commutation Table do not exactly correspond with those which may be deduced from the abridged table referred to. This table states the survivors of 1,000 births to be, at the ages of 6 and 7, for instance, 919 and 912 respectively; while the use we have made of the data enables us to say, that the survivors at these ages, out of 100,000,000 of births, will be 91,912,811 and 91,239,410. As regards the correctness of our Commutation Table, it has been subjected to the severest tests in the way of verification, and we are confident that no error of any consequence will be found in it. Mr. Finlaison, in his Report, gives the values of annuities at all ages, at 4 per cent., as deduced from his data by the ordinary method, and carried to seven places, which are two more than are usually given. The values derived from our table will be found to correspond with those to the last place. It has been thought proper to make these remarks as to the degree of confidence that may be reposed in our table, in case any one should feel disposed to apply it to practical purposes.

We shall now explain the notation, as regards the rates of interest and mortality, which we shall employ in our demonstrations.

It is shown by writers on interest, and, indeed, in most elementary works on algebra, that if r represent the interest of £1 for a year, then will

$1+r$	be the *amount* of	£1	in	1	year,
$(1+r)^2$	,, ,,	£1	,,	2	years,
$(1+r)^3$	,, ,,	£1	,,	3	,,
$(1+r)^4$	,, ,,	£1	,,	4	,,

and, generally, $(1+r)^x$ will be the amount of £1 in x years, where x may represent any number whatsoever, all at compound interest. It is also shown that unity divided by the amount of £1 in a given time is equal to the *present value* of £1 to be received at the end of that time—that is, to the sum which would, in the given time, just amount, also at compound interest, to £1. Hence—

$\frac{1}{1+r}$	is the present value of	£1	due at the end of	1 year,
$\frac{1}{(1+r)^2}$	,, ,,	£1	,,	2 years,
$\frac{1}{(1+r)^3}$	,, ,,	£1	,,	3 ,,

and, generally, $\frac{1}{(1+r)^x}$ is the present value of £1 due at the end of x years—x, as before, denoting any number whatsoever.

It is usual to denote $\frac{1}{1+r}$ by v. Hence the foregoing present values will be more conveniently represented by $v, v^2, v^3 \ldots v^x$, respectively. It is obvious that the present value of any other sum will be found by multiplying the present value of £1, due at the end of the same number of years, by the number of pounds in that sum. Copious tables of the present values of £1 due at the end of any number of years, from 1 to 100, together with their logarithms, are given in Mr. Jones's work, Part I.

The indications of the mortality table are represented as follows:—the number shown by the table to attain to any age is denoted by the letter l, with the age attached as a suffix. Thus, l_0, l_1, l_2, l_3, l_x, &c., denote the numbers who attain the successive ages, 0, 1, 2, 3, x, &c., respectively. These symbols will, of course, denote different numbers in the case of different tables. Thus, l_0, representing the number alive at birth, which is called the *radix* of the table, will, in the case of the Carlisle Table, be equal to 10,000, and in that of the Northampton Table, to 11,650; and the values of the other symbols will in like manner vary. But the great advantage of the use of symbols is, that we have no need to distract ourselves with their particular values until we reach the final solution of the problem with which we may be engaged. Also, since the number who die in any year of age—that is, who enter upon that year and do not live to complete it—is equal to the difference between the number who complete that year and the number who completed the previous year, these decrements, as they are called, will be represented as follows:—

l_0-l_1	is the number	who die in their	1st year,
l_1-l_2	,,	,,	2nd ,,
l_2-l_3	,,	,,	3rd ,,
$l_{x-1}-l_x$	,,	,,	xth ,,
l_x-l_{x+1}	,,	,,	$(x+1)$th year,

and so on, x representing any age whatsoever.

We are now in a condition to enter upon a description of the Commutation Tables, and their construction, and to show the algebraical properties that belong to them in virtue of that construction. We refer, for illustration, to the table contained in the two following pages.

The table consists, it will be seen, of two side columns con-

Commutation Table—Single Male Life.

Government Rate of Mortality—Interest 4 per Cent.

Age.	D.	N.	S.	M.	R.	Age.
0	10000·0000	189251·4911	3483123·3173	2336·48113	57621·69308	0
1	9431·1674	179820·3237	3293871·8262	2152·26386	55285·21195	1
2	8905·8830	170914·4407	3114051·5025	1989·71684	53132·94809	2
3	8434·3219	162480·1188	2943137·0618	1860·68958	51143·23125	3
4	8008·6623	154471·4565	2780656·9430	1759·42701	49282·54167	4
5	7620·6994	146850·7571	2626185·4865	1679·48959	47523·11466	5
6	7264·0029	139586·7542	2479334·7294	1615·89696	45843·62507	6
7	6933·4451	132653·3091	2339747·9752	1564·72403	44227·72811	7
8	6623·2532	126030·0559	2207094·6661	1521·20284	42663·00408	8
9	6330·0454	119700·0105	2081064·6102	1482·73560	41141·80124	9
10	6051·0605	113648·9500	1961364·5997	1447·21386	39659·06564	10
11	5785·5704	107863·3796	1847715·6497	1414·45703	38211·85178	11
12	5532·5360	102330·8436	1739852·2701	1383·94450	36797·39475	12
13	5291·3245	97039·5191	1637521·4265	1355·52288	35413·45025	13
14	5061·0571	91978·4620	1540481·9074	1328·76796	34057·92737	14
15	4839·8540	87138·6080	1448503·4454	1302·22087	32729·15941	15
16	4626·0186	82512·5894	1361364·8374	1274·53377	31426·93854	16
17	4417·5780	78095·0114	1278852·2480	1244·01695	30152·40477	17
18	4213·8385	73881·1729	1200757·2366	1210·18431	28908·38782	18
19	4014·0326	69867·1403	1126876·0637	1172·44914	27698·20351	19
20	3818·9594	66048·1809	1057008·9234	1131·76185	26525·75437	20
21	3629·2895	62418·8914	990960·7425	1088·97492	25393·99252	21
22	3442·7376	58976·1538	928541·8511	1042·01100	24305·01760	22
23	3264·0617	55712·0921	869565·6973	995·74796	23263·00660	23
24	3094·1061	52617·9860	813853·6052	951·33340	22267·25864	24
25	2932·9909	49684·9951	761235·6192	909·22226	21315·92524	25
26	2780·7540	46904·2411	711550·6241	869·79274	20406·70298	26
27	2639·0772	44265·1639	664646·3830	835·06793	19536·91024	27
28	2505·0301	41760·1338	620381·2191	802·52378	18701·84231	28
29	2378·0529	39382·0809	578621·0853	771·89394	17899·31853	29
30	2257·6521	37124·4288	539239·0044	742 95673	17127·42459	30
31	2143·4127	34981·0161	502114·5756	715·55005	16384·46786	31
32	2034·9211	32946·0950	467133·5595	689·49752	15668·91781	32
33	1932·0753	31014·0197	434187·4645	664·91786	14979·42029	33
34	1834·6160	29179·4037	403173·4448	641·76920	14314·50243	34
35	1742·1788	27437·2249	373994·0411	619·89412	13672·73323	35
36	1654·3493	25782·8756	346556·8162	599·07143	13052·83911	36
37	1570·6596	24212·2160	320773·9406	579·01055	12453·76768	37
38	1490·8889	22721·3271	296561·7246	559·64988	11874·75713	38
39	1414·7080	21306·6191	273840·3975	540·81084	11315·10725	39
40	1342·1355	19964·4836	252533·7784	522·65027	10774·29641	40
41	1273·0650	18691·4186	232569·2948	505·20029	10251·64614	41
42	1207·7753	17483·6433	213877·8762	488·87461	9746·44585	42
43	1145·6257	16338·0176	196394·2329	473·17792	9257·57124	43
44	1086·6545	15251·3631	180056·2153	458·26918	8784·39332	44
45	1030·6715	14220·6916	164804·8522	444·08057	8326·12414	45
46	977·4798	13243·2118	150584·1606	430·53017	7882·04357	46
47	927·3068	12315·9050	137340·9488	417·95256	7451·51340	47
48	879·4023	11436·5027	125025·0438	405·71372	7033·56084	48
49	833·5711	10602·9316	113588·5411	393·70562	6627·84712	49

Commutation Table (continued).

Age.	D.	N.	S.	M.	R.	Age.
50	789·5004	9813·4312	102985·6095	381·69530	6234·14150	50
51	746·9195	9066·5117	93172·1783	369·47976	5852·44620	51
52	705·1776	8361·3341	84105·6666	356·46560	5482·96644	52
53	664·7128	7696·6213	75744·3325	343·123[illegible]4	5126·50084	53
54	625·4090	7071·2123	68047·7112	329·38523	4783·37780	54
55	587·3514	6483·8609	60976·4989	315·38164	4453·99257	55
56	550·6411	5933·2198	54492·6380	301·26191	4138·61093	56
57	515·2537	5417·9661	48559·4182	287·05295	3837·34902	57
58	481·3810	4936·5851	43141·4521	272·99761	3550·29607	58
59	449·1615	4487·4236	38204·8670	259·29288	3277·29846	59
60	418·5753	4068·8483	33717·4434	245·98204	3018·00558	60
61	389·8164	3679·0319	29648·5951	233·32228	2772·02354	61
62	362·7862	3316·2457	25969·5632	221·28490	2538·70126	62
63	337·2557	2978·9900	22653·3175	209·70782	2317·41626	63
64	312·7669	2666·2231	19674·3275	198·19033	2107·70844	64
65	289·2939	2376·9292	17008·1044	186·74680	1909·51812	65
66	266·8385	2110·0907	14631·1752	175·41815	1722·77132	66
67	244·8012	1865·2895	12521·0845	163·64386	1547·35317	67
68	223·8080	1641·4815	10655·7950	152·06610	1383·70931	68
69	203·8598	1437·6217	9014·3135	140·72583	1231·64321	69
70	184·9140	1252·7077	7576·6918	129·62088	1090·91737	70
71	166·9336	1085·7741	6323·9841	118·75251	961·29649	71
72	149·9696	935·8045	5238·2100	108·20904	842·54398	72
73	134·1890	801·6155	4302·4055	98·19656	734·33495	73
74	119·7040	681·9115	3500·7900	88·87269	636·13839	74
75	106·3778	575·53365	2818·87847	80·15044	547·26570	75
76	94·11924	481·41441	2243·34482	71·98333	467·11526	76
77	83·30894	398·10547	1761·93041	64·79300	395·13193	77
78	73·11608	324·98939	1363·82494	57·80433	330·33893	78
79	63·41526	261·57413	1038·83555	50·91567	272·53460	79
80	54·36266	207·21147	777·26142	44·30211	221·61892	80
81	45·97142	161·24005	570·04995	38·00176	177·31681	81
82	38·23230	120·00775	408·80990	32·03076	139·31505	82
83	31·33915	91·66860	285·80215	26·60808	107·28429	83
84	25·24443	66·42417	194·13355	21·71872	80·67621	84
85	19·83120	46·59297	127·70938	17·27643	58·95749	85
86	15·13649	31·45648	81·11641	13·34445	41·68106	86
87	11·05347	20·403011	49·659934	9·843603	28·336614	87
88	7·702274	12·700737	29·256923	6·917542	18·493011	88
89	5·113615	7·587122	16·556186	4·625125	11·575469	89
90	3·249834	4·337288	8·969064	2·958021	6·950344	90
91	1·946549	2·390739	4·631776	1·779730	3·992323	91
92	1·152956	1·237783	2·241037	1·061004	2·212593	92
93	·6407773	·5970057	1·003254	·5931702	1·1515892	93
94	·3327112	·2642945	·4062485	·3097495	·5584190	94
95	·1605972	·1036973	·1419540	·1504320	·2486695	95
96	·0716510	·0320462	·0382567	·0676627	·0982375	96
97	·0258357	·0062105	·0062105	·0246032	·0305748	97
98	·0062105	·0000000	·0000000	·0059716	·0059716	98

taining the ages, and five inner columns, headed D, N, S, M, and R, respectively. There is another column, headed C, which is essential to the theory of the tables, and the proper place for which, if inserted, would be between columns S and M. But as this column is not required in practice, it is never exhibited. The letters D, N, S, &c., are chosen quite arbitrarily, and have no reference to the signification of the columns at the head of which they are placed.

The number in any column, opposite to any age, is denoted by the letter at the head of the column, with the age attached as a suffix. Thus, the number corresponding to age 20 in column D, for example, is denoted by D_{20}, in column N by N_{20}, and so of the other columns. If the age be x, the corresponding numbers are denoted by D_x, N_x, &c.

The columns are constructed as follows:—Take any age, 20, for example. The number in column D, opposite age 20, is equal to the product of the number represented by the mortality table to attain the age 20, into the present value of £1 due at the end of 20 years. So that the algebraical expression will be $D_{20}=l_{20}v^{20}$.

And, generally, the number corresponding to any age in column D is equal to the number who complete that year of their age multiplied by the present value of £1 due at the end of as many years as are equal to the age. Hence, the general expression for the value of the numbers in column D will be $D_x=l_xv^x$,* x denoting any age.

Hence also, if the age be 0, we have $D_0=l_0v^0=l^0$, since $v_0=1$. That is, the first number in column D is equal to the radix of the mortality table.

Column N is formed from column D, by inserting opposite each age in N the sum of the numbers opposite all the higher ages in D. For example—

$$N_{20}=D_{21}+D_{22}+D_{23}+\&c., \text{ to the end.}$$
$$N_{21}=D_{22}+D_{23}+D_{24}+\&c.$$

So that the expression for the general term of this column is

$$N_x=D_{x+1}+D_{x+2}+D_{x+3}+\&c.$$

x, as before, denoting any age.

* By an expression for the general term of a series, is meant, an expression in which a variable quantity is introduced, and which, by giving any particular value to the variable, gives the term of the series corresponding to that value. Thus, in the above general expression, $D_x=l_xv^x$, x, which denotes the age, is the variable; and if we give to it a particular value, we have immediately the term of the series corresponding to that value. For instance, if $x=20$, the expression becomes

$$D_{20}=l_{20}v^{20},$$

and this is the value in column D corresponding to age 20.

It follows from this that the last N in the table is 0. For, since $N_x = D_{x+1} + \ldots$, if x be the highest age, D_{x+1} and all the following terms vanish, since there are no survivors at the ages denoted by $x+1$, &c. Also, the first N in the table is equal to the sum of all the D's except the first, or $N_0 = D_1 + D_2 + D_3 + \ldots$

Column S is formed by inserting in it, opposite each age, the sum of the numbers in N opposite that age and all the higher ages. The expression for the general term is, therefore—

$$S_x = N_x + N_{x+1} + N_{x+2} + \&c.$$

And it differs from D in including, at each age, the number corresponding to that age in the preceding column. It follows from this that the last S is 0, and that the first S is equal to the sum of all the numbers in column N.

The three columns just described are called the *annuity* columns. The remaining three are called the *assurance* columns. The last named we now proceed to describe.

Column C is formed by inserting, opposite each age, the product of the number who die in the following year of their age by the present value of £1 to be received at the end of a number of years, which is equal to one more than the age in question. Thus—

$$C_0 = (l_0 - l_1)v,$$
$$C_1 = (l_1 - l_2)v^2,$$

and so on. And the expression for the general term will be

$$C_x = (l_x - l_{x+1})v^{x+1}.$$

Column C, as already remarked, is not shown in the table. It is used only for the construction of column M; and M is derived from it precisely as S was derived from N. So that the expression for the general term of this column is

$$M_x = C_x + C_{x+1} + C_{x+2} + \&c.$$

Column R, also, is formed from M, in exactly the same manner as M was formed from C; and hence the expression for its general term is

$$R_x = M_x + M_{x+1} + M_{x+2} + \&c.^*$$

We proceed now to demonstrate a few of the algebraical properties, and relations amongst each other, that belong to the

* It will be afterwards seen that there is no necessity for employing those particular powers of v, which have been made use of in the construction of the table. The only condition as to these powers, which is indispensable to the possession by the table of the required properties, is, that their indices shall form an increasing arithmetical series, of which the common difference is unity. The powers that have been chosen possess the advantage of imparting to the expressions for the values of the numbers in columns D and C, a symmetry that would not otherwise have belonged to them.

numbers in the different columns. These will be useful for after reference, with a view to which we shall number them as we go along.

By algebraical properties and relations, we mean those properties and relations that subsist in virtue of the mode in which the table has been constructed, without reference to what the numbers employed in that construction denote.

Among the algebraical relations must be classed those which we have just seen to subsist; which, therefore, we here repeat, for the convenience of reference.

$$\left.\begin{array}{l} D_x = l_x v^x \\ C_x = (l_x - l_{x+1}) v^{x+1} \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{l} N_x = D_{x+1} + D_{x+2} + D_{x+3} + \&c. \\ S_x = N_x + N_{x+1} + N_{x+2} + \&c. \\ M_x = C_x + C_{x+1} + C_{x+2} + \&c. \\ R_x = M_x + M_{x+1} + M_{x+2} + \&c. \end{array}\right\} \quad (2)$$

In what follows we call x the present age; and since, in the above expression, x denotes any age, we may substitute for it $x+n$; the only limitation* that we make with regard to n, being, that it shall not exceed the difference between x and the oldest age in the table, so that $x+n$ may not exceed that age. Making this substitution, then, the above equations will still hold, and we shall have

$$\left.\begin{array}{l} D_{x+n} = l_{x+n} v^{x+n} \\ C_{x+n} = (l_{x+n} - l_{x+n+1}) v^{x+n+1} \end{array}\right\} \quad (3)$$

$$\left.\begin{array}{l} N_{x+n} = D_{x+n+1} + D_{x+n+2} + \&c. \\ S_{x+n} = N_{x+n} + N_{x+n+1} + \&c. \\ M_{x+n} = C_{x+n} + C_{x+n+1} + \&c. \\ R_{x+n} = M_{x+n} + M_{x+n+1} + \&c. \end{array}\right\} \quad (4)$$

It thus appears that the nth term from the present age, in columns S, M, and R, expresses *the sum of the* n*th and following terms* from the same age in columns N, C, and M respectively; and that the nth term from the present age, in column N, expresses *the sum of all the terms after the* nth from that age in column D. If we wish *the sum of the* n*th and following terms* in D, we have only to write $x-1$ for x in the first of the expressions (4), which then becomes

$$N_{x+n-1} = D_{x+n} + D_{x+n+1} + \&c. \quad (5)$$

And, generally, it will be obvious that, to make our summation commence 1, 2, 3, &c., terms earlier or later, we have only to substitute, in the expressions (4), in the former case $(x-1)$, $(x-2)$, $(x-3)$, &c., and in the latter $(x+1)$, $(x+2)$, $(x+3)$, &c., for x.

* Algebraically considered, it is not necessary to make this limitation.

The expressions (2) and (4) enable us to find the sum of the first n terms, either after or commencing with the present age, in the columns D, N, C, and M; for we have only to subtract from the sum of *all* the terms in these columns, after or commencing with the present age, the sum of those after or commencing with the nth from that age. Hence the sum of the first n terms, *after* the present age,

$$\left.\begin{array}{lll} \text{in D is} & N_x - N_{x+n} & \\ \text{N} & S_{x+1} - S_{x+n+1} & \\ \text{C} & M_{x+1} - M_{x+n+1} & \\ \text{M} & S_{x+1} - S_{x+n+1} & \end{array}\right\} \quad (6)$$

And the sum of the first n terms, *commencing with* the present age,

$$\left.\begin{array}{lll} \text{in D is} & N_{x-1} - N_{x+n-1} & \\ \text{N} & S_x - S_{x+n} & \\ \text{C} & M_x - M_{x+n} & \\ \text{M} & R_x - R_{x+n} & \end{array}\right\} \quad (7)$$

Also, according to the principle laid down above, if, in the expressions (6) and (7), we substitute $x+k$ for x, we are furnished with expressions for the sum of the first n terms, either after or commencing with the $(x+k)$th.

Thus, the sum of the first n terms, *after* the $(x+k)$th,

$$\left.\begin{array}{lll} \text{in D is} & N_{x+k} - N_{x+k+n} & \\ \text{N} & S_{x+k+1} - S_{x+k+n+1} & \\ \text{C} & M_{x+k+1} - M_{x+k+n+1} & \\ \text{M} & R_{x+k+1} - R_{x+k+n+1} & \end{array}\right\} \quad (8)$$

And the sum of the first n terms, *commencing with* the $(x+k)$th,

$$\left.\begin{array}{lll} \text{in D is} & N_{x+k-1} - N_{x+k+n-1} & \\ \text{N} & S_{x+k} - S_{x+k+n} & \\ \text{C} & M_{x+k} - M_{x+k+n} & \\ \text{M} & R_{x+k} - R_{x+k+n} & \end{array}\right\} \quad (9)$$

If, in the expressions (7), we suppose $n=1$, they become respectively—

$$\left.\begin{array}{l} D_x = N_{x-1} - N_x \\ N_x = S_x - S_{x+1} \\ C_x = M_x - M_{x+1} \\ M_x = R_x - R_{x+1} \end{array}\right\} \quad (10)$$

If, in the expressions (10), we substitute $x+1$ for x, we obtain the same expressions as we should obtain by making n in (6)$=1$. And, in like manner, by substituting in (10) $x+k$, and $x+k+1$ for x, we obtain the same expressions as we should obtain from (9) and (8) respectively, by making $n=1$ in these expressions. The

expressions just referred to, as well as others which may be derived by transposition from those we have given, we do not think it necessary to set down, since their mode of derivation is so extremely simple; and we shall accordingly refer, as we have occasion for them, to the expressions which we have shown to contain them.

Again, since

$$\begin{aligned} N_x &= D_{x+1} + D_{x+2} + D_{x+3} + D_{x+4} + \&c. \\ N_{x+1} &= \qquad\quad D_{x+2} + D_{x+3} + D_{x+4} + \&c. \\ N_{x+2} &= \qquad\qquad\qquad D_{x+3} + D_{x+4} + \&c. \end{aligned}$$

and so on. If we add these equations together, observing that

$$N_x + N_{x+1} + N_{x+2} + \&c. = S_x,$$

the resulting equation will be

$$S_x = D_{x+1} + 2D_{x+2} + 3D_{x+3} + 4D_{x+4} + \&c. \quad . \quad (11)$$

And, in the same way, we should obtain

$$R_x = C_x + 2C_{x+1} + 3C_{x+2} + 4C_{x+3} + \&c. \quad . \quad (11)$$

Sometimes, in the Commutation Tables, the assurance columns are not exhibited. We therefore proceed to show how their place may be supplied by means of the annuity columns.

$$C_x = (l_x - l_{x+1})\, v^{x+1} = l_x v^{x+1} - l_{x+1} v^{x+1};$$

but

$$l_x v^{x+1} = v D_x \text{ and } l_{x+1} v^{x+1} = D_{x+1};$$

consequently,

$$C_x = vD_x - D_{x+1} \quad . \quad . \quad . \quad . \quad (12)$$

Again, since

$$\begin{aligned} C_x &= vD_x - D_{x+1} \\ C_{x+1} &= vD_{x+1} - D_{x+2} \\ C_{x+2} &= vD_{x+2} - D_{x+3} \\ C_{x+3} &= vD_{x+3} - D_{x+4} \end{aligned}$$

and so on, to the end of life. If we add these equations, observing that

$$C_x + C_{x+1} + C_{x+2} + \&c. = M_x,$$

$$vD_x + vD_{x+1} + \&c. = v(D_x + D_{x+1} + \&c.) = vN_{x-1},$$

$$\text{and } D_{x+1} + D_{x+2} + \&c. = N_x,$$

the resulting equation will be

$$M_x = vN_{x-1} - N_x \quad . \quad . \quad . \quad . \quad (13)$$

The analogous expression for R, in terms of S, and which is obtained in precisely the same way, is

$$R_x = vS_{x-1} - S_x \quad . \quad . \quad . \quad . \quad (13)$$

The expressions just deduced may be exhibited in a somewhat different form, which is, in certain circumstances, more convenient than the other.

Since, by (10),

$$N_x = N_{x-1} - D_x,$$

the first of the foregoing expressions becomes, by substitution,

$$\begin{aligned} M_x &= vN_{x-1} - N_{x-1} + D_x \\ &= D_x - N_{x-1} + vN_{x-1} \\ &= D_x - (1-v)N_{x-1} \quad . \quad . \quad . \quad (14) \end{aligned}$$

The corresponding expression for R_x, obtained in the same way, is

$$R_x = N_{x-1} - (1-v)S_{x-1} \quad . \quad . \quad (14)$$

Of the foregoing expressions, those numbered (1) to (11) are extensively useful, in deducing and simplifying the formulæ in the practical application of the tables; and the remaining expressions, numbered (12) to (14), besides their use in supplying the place of the assurance columns, when required, serve also to verify the tables. We have not space here to give examples illustrative of any of the foregoing formulæ. But these are so very simple, that this does not seem at all necessary.

For the convenience of those who may wish to test the table on pp. 90, 91, by means of the formulæ of verification given above, we subjoin the following elements:—

When the rate of interest is 4 per cent (as in our table),

$$v = \frac{1}{1{\cdot}04} = {\cdot}96153846,$$

and its logarithm is ·98296666.

Also $1 - v = {\cdot}03846154$, and its logarithm is ·58502665.

The formulæ we have demonstrated may appear to some, at first view, both complex and confused. They are, nevertheless, eminently remarkable for their simplicity and symmetry. This, a further and practical acquaintance with them will make abundantly manifest.

We now proceed to show how the present values of benefits may be deduced from the Commutation Tables; but we must first explain what is meant by the present value of a benefit.

As regards the purchaser of a single benefit, it is seldom indeed that he will receive the precise value (using the word in its ordinary sense) of what he has paid. In the case of certain kinds of benefits, indeed, it is impossible that there should, in any case, be an exact

compensation in this sense. For instance, if the benefit be an endowment—that is, a sum of money to be received in the event of the purchaser attaining a certain specified age; if he do not attain that age, neither he nor his representatives will receive anything, and, consequently, the money he has paid will be lost. On the other hand, if he do attain the specified age, the sum to be then received will be more than his payment would have amounted to if it had been put out at interest at the time the bargain was made; else he would not have run the risk of losing it in the interval. In the case of such a benefit, therefore, the purchaser *must* be either a gainer or a loser. If the benefit consist of an annuity, the purchaser *may* live just such a period as that the number of payments he will receive will be exactly equivalent to the money he has paid. If he live a shorter time he will be a loser, and if a longer he will be a gainer. Similar remarks apply to the case of a life assurance: the purchaser *may* die, and the assurance be received at the end of a period, the improvement of his premium or premiums during which would have produced an amount just equal to the assurance. But as it is much more likely that he will die either before or after the period of this equality, so the probability is proportionally greater that he or his representatives will be either gainers or losers.

But while there is this uncertainty in individual cases, there is, in a sufficiently numerous aggregation of cases, a regularity in the occurrences of the events on which the payments of benefits are usually made to depend, which admits of the calculation of the *average* values of those benefits to a very considerable degree of nicety. For, if we assume the mortality which will be experienced by a class of purchasers, sufficiently numerous to secure an average, to be the same as that indicated in the table which we take as our guide (and which table presents the averages of a number of observations), we say, making this assumption, it is evidently quite practicable to name the sum which will require to be contributed by the whole body of purchasers, to afford to each of them a benefit of a certain amount, on the occurrence of a specified contingency, the time or the fact of this occurrence, or both, being uncertain *in each individual case.* And this sum, divided by the number of purchasers, will be the amount which each ought to contribute, since all at the time of purchase are supposed to be equally likely to come into possession of the benefit. It is the last-named amount, obtained in this manner, which is called the *present value,* or *single premium,* of the benefit to be purchased.

The meaning of the term "present value," might, perhaps, have been more briefly explained, by defining it as the sum which would be required from each of a large number of purchasers of benefits of the same kind, so that the seller should be in the end neither a gainer nor a loser.

We now proceed to the application of the tables. But first, we remark, once for all, that in what follows we always suppose the *amount* of the benefit to be purchased to be £1; except, of course, in the cases of an increasing annuity or assurance. In these we suppose that amount only, to which the purchaser becomes or may become first entitled, to be £1. When the present value of a benefit of £1 on any life is found, that of a similar benefit of any other amount, on the same life, will obviously be found by multiplying this present value by the number of pounds in that amount.

PROBLEM I.—To find the present value of an endowment of £1 —that is, of £1 to be received at the end of n years provided (x)* be then alive.

The number of individuals now alive, of the given age, according to the mortality table, is l_x; and the survivors of these, at the end of n years, is the number represented by the table to attain the age $x+n$, viz., l_{x+n}. This number is, consequently, the number of pounds which will have to be paid at the end of n years. But, since money makes interest, the present value of this sum, or $l_{x+n}v^n$, contributed now, will be sufficient to provide for the payment of the benefit. And, since all now alive contribute equally, the amount to be contributed by each—that is, the required present value—will be $\frac{l_{x+n}v^n}{l_x}$.†

To adapt this expression to the Commutation Tables, multiply the numerator and denominator by v^x (which will not alter its value), and it becomes $\frac{l_{x+n}v^{x+n}}{l_x v^x}$.

But the numerator of this expression is equal to D_{x+n}, and

* By this symbol is meant, and the phrase may in reading be substituted for it, "a life now aged x years."

† We have assumed above that the number of purchasers of endowments will be the number represented by the mortality table to be alive at the age at which the purchase is made; but the only assumption as to their number which it is *necessary* to make is, that this number will be sufficient to secure an average mortality proportional to that represented in the table. If we had assumed any other number, we should have had to find, by a proportion, the number of survivors at the advanced age. But, by the assumption in the text, this operation is saved. In both cases the result would be the same. The value of a fraction depends, not on the *absolute magnitude* of its terms, but on their *relative magnitude* or *ratio*. The value of 3-4ths is the same as that of 6-8ths or 9-12ths, because the ratio of 3 to 4 is equal to that of 6 to 8 or 9 to 12.

The same remarks will apply to the other benefits.

the denominator to D_x, by (1) and (3). Hence the expression becomes finally $\frac{D_{x+n}}{D_x}$.

From this general expression we obtain at once the solution of any particular case that may be proposed, by giving to x and n the proper values. Thus, if $x=10$ and $n=11$, that is, if the endowment is to be paid on (10) attaining the age of 21, the expression becomes $\frac{D_{21}}{D_{10}}$; and the numerical solution is effected by taking the proper numbers from the table, and operating upon them as indicated by the formula.

If $n=0$, that is, if the endowment is to be paid immediately, then we have, whatever the age may be, $\frac{D_x}{D_x}=1$, that is, to £1, as we evidently ought to have.

Example 1. $x=10$, $n=11$.

$$\frac{D_{21}}{D_{10}}=\frac{3629{\cdot}2895}{6051{\cdot}0605}={\cdot}59978=12s.$$

And if the amount to be received were £100, the present value would be $\cdot59978 \times 100 = 59\cdot978 =$ £59. 19*s.* 7*d.*

Example 2. $x=18$, $n=6$.

$$\frac{D_{24}}{D_{18}}=\frac{3094{\cdot}1061}{4213{\cdot}8385}={\cdot}73427=14s.\ 8d.$$

For £100, the present value would be $\cdot73427 \times 100 = 73\cdot427$ = £73. 8*s.* 6*d.*

PROBLEM II.—To find the present value of a life annuity of £1 on (x).

This benefit consists of a series of payments of £1, to be made at the end of 1, 2, 3, &c., years, to the end of life. Its present value, therefore, will be the sum of the present values of the several payments. These present values are, by last problem,

$$\frac{D_{x+1}}{D_x},\ \frac{D_{x+2}}{D_x},\ \frac{D_{x+3}}{D_x},$$

and so on; and their sum is

$$\frac{D_{x+1}+D_{x+2}+D_{x+3}+\&c.}{D_x},$$

which, by (1), is equal to $\frac{N_x}{D_x}$, which is, therefore, the present value required.

Example 1. $x=20$.

$$\frac{N_{20}}{D_{20}}=\frac{66048\cdot1809}{3818\cdot9594}=17\cdot2948=\pounds17.\ 5s.\ 11d.$$

Example 2. $x=60$.

$$\frac{N_{60}}{D_{60}}=\frac{4068\cdot8483}{418\cdot5753}=9\cdot7207=\pounds9.\ 14s.\ 5d.$$

If the annuity be *due*, that is, if the first payment is to be made immediately, then the present value of that payment, that is, $\frac{D_x}{D_x}$, must be added to the above expression, which then becomes

$$\frac{D_x+N_x}{D_x}=\frac{N_{x-1}}{D_x} \quad . \quad . \quad . \quad (10)$$

The same result is obtained by commencing the summation of the series in the numerator of the first expression a year earlier, that is, with D_x, instead of D_{x+1}.

Example.—Required the present value of an annuity *due* of £1 on (20), that is $x=20$.

$$\frac{N_{19}}{D_{20}}=\frac{69867\cdot1403}{3818\cdot9594}=18\cdot2948=\pounds18.\ 5s.\ 11d.$$

Problem III.—To find the present value of an annuity of £1 deferred for n years on (x).

This benefit consists of a series of payments of £1, to be made $n+1$, $n+2$, $n+3$, &c., years hence, to the end of life; and its present value will, therefore, be the sum of the present values of these payments. These present values are found by Problem I., and their sum is

$$\frac{D_{x+n+1}+D_{x+n+2}+D_{x+n+3}+\&c.}{D_x},$$

which, by (4), is equal to $\frac{N_{x+n}}{D_x}$, and this is, therefore, the present value required.

Example.—If $x=20$, and $n=10$, that is, if the annuity is to be entered upon by a person now aged 20, when he attains the age of 30, its present value will be

$$\frac{N_{30}}{D_{20}}=\frac{37124\cdot4288}{3818\cdot9594}=9\cdot7211=\pounds9.\ 14s.\ 5d.$$

If the first payment is to be made n years hence, the requisite

change will be made in the formula, as in last problem, by writing $x-1$ for x in the numerator. The expression for the present value in this case, therefore, will be $\frac{N_{x+n-1}}{D_x}$, which is also, as it evidently ought to be, the expression for the present value of an annuity deferred for $n-1$ years.

If, in the expression $\frac{N_{x+n}}{D_x}$, we suppose n, the period of deferment, equal to 0—that is, that the annuity is to be entered upon immediately—the expression becomes $\frac{N_x}{D_x}$, the same as in last problem, which problem is, therefore, but a particular case of the present; and the following rule will apply to both :—

Divide the number in column N, opposite the age at which the annuity is to be entered upon, by the number in column D, opposite the present age; the quotient will be the present value of the annuity.

A similar remark will apply to the other benefits, and a corresponding rule for all of them might be formed. But, as our space is limited, the present intimation must suffice.

PROBLEM IV.—To find the present value of a temporary annuity of £1 for the next n years, on (x).

This benefit consists of n payments of £1, to be made at the end of 1, 2, 3, ... n years, if x be then alive. The present values of these payments are

$$\frac{D_{x+1}}{D_x},\ \frac{D_{x+2}}{D_x},\ \frac{D_{x+3}}{D_x},\ \dots\ \frac{D_{x+n}}{D_x},$$

the sum of which is

$$\frac{D_{x+1}+D_{x+2}+D_{x+3}+\ \dots\ D_{x+n}}{D_x};$$

and this, by (6), is equal to

$$\frac{N_x-N_{x+n}}{D_x},$$

which is, therefore, the present value of the annuity.

This formula might also have been deduced by subtracting from the present value of an annuity for the whole life, that of an annuity deferred for n years, since the present value of an annuity during the next n years, and that of an annuity to commence at the end of n years, are evidently together equal to the present value of an annuity for the whole life. Thus, the present value of

the life annuity is (Prob. II.) $\frac{N_x}{D_x}$, and that of the deferred annuity is (Prob. III.) $\frac{N_{x+n}}{D_x}$. Their difference, or $\frac{N_x - N_{x+n}}{D_x}$, is the present value of the temporary annuity, as before.

Example.—Required the present value of an annuity of £1 during the next 10 years, on (20).

$$\textit{Answer.}\ \frac{N_{20}-N_{30}}{D_{20}} = \frac{66048{\cdot}1809 - 37124{\cdot}4288}{3818{\cdot}9594} = \frac{28923{\cdot}7521}{3818{\cdot}9594}$$
$$= 7{\cdot}5737 = £7.\ 11s.\ 6d.$$

If this be added to 9·7211, which we found by last problem to be the value of an annuity deferred for 10 years, on a life of the same age, the sum 17·2948 will be the value of an annuity on the whole life; and it corresponds with the value which we deduced for that age by Problem II.

Here we may repeat, once for all, as the remark applies equally to the expressions for all the benefits, that so long as the payments continue the same in number, if they commence a year earlier than we have supposed, the change requisite to be made in the formula is the substitution of $x-1$ for x in the numerator. The above expression thus becomes

$$\frac{N_{x-1} - N_{x+n-1}}{D_x}.$$

PROBLEM V.—To find the present value of a deferred temporary annuity of £1 on (x), the period of deferment being k years, and that of continuance n years.

This benefit consists of n payments of £1, to be made at the end of $k+1$, $k+2$, $k+3$, ... $k+n$ years. The present values of the payments are (Prob. I.)

$$\frac{D_{x+k+1}}{D_x},\ \frac{D_{x+k+2}}{D_x},\ \ldots\ \frac{D_{x+k+n}}{D_x};$$

and the sum of these expressions is

$$\frac{D_{x+k+1} + D_{x+k+2} + \ldots\ D_{x+k+n}}{D_x},$$

which, by (8), is equal to

$$\frac{N_{x+k} - N_{x+k+n}}{D_x};$$

and this is evidently the present value required.

It will be seen that this expression may be derived from that in the last problem, by changing x into $x+k$ or n into $k+n$ in the numerator.

Example.—Required the present value of an annuity of £1 on (20), to be entered upon 10 years hence, and to continue 10 years.

Here $x=20$, $k=10$, and $n=10$.

$$\frac{N_{30}-N_{40}}{D_{20}}=\frac{37124 \cdot 4288-19964 \cdot 4836}{3818 \cdot 9594}=\frac{17159 \cdot 9452}{3818 \cdot 9594}=4 \cdot 4934$$

$$=£4.\ 9s.\ 10d.$$

(*To be continued.*)

On the Value of Mr. Gompertz's Formula for the Number Living, in Terms of the Mortality according to Age, compared with the Value of a similar Formula published in 1832. *By* T. R. EDMONDS, B.A., *formerly of Trinity College, Cambridge.*

IN the *Assurance Magazine* (Oct., 1860, and July, 1861) there have appeared two papers written by me to show in what respect the law of human mortality described by Mr. Gompertz in 1825 differs from the law described by me in 1832. In the same papers I have also shown in what respect the formula of Mr. Gompertz, $y=dg^{p^x}$, for the number living in terms of the mortality according to age, differs from my formula for the same purpose, which is $y=10^{\frac{k^2a}{\lambda p}(1-p^x)}$ or $y=g^{1-p^x}$, on putting $g=10^{+\frac{k^2a}{\lambda p}}$ instead of $g=10^{-\frac{k^2a}{\lambda p}}$, as assumed by Mr. Gompertz.

In the *Assurance Magazine* (April, 1861, and Oct., 1861) there have also appeared two papers, in opposition to mine, which have been written by Mr. Sprague, the accepted advocate of Mr. Gompertz. In the latter paper of Mr. Sprague the opposition has been confined chiefly to the question of the relative value of Mr. Gompertz's formula and my formula of mortality. As this question appears to me to be the only one which can, with any apparently good reason, become the subject of dispute between Mr. Gompertz and myself, I will follow Mr. Sprague's example, and confine my coming remarks chiefly to the comparative merits of the two competing formulæ.

The formulæ to be compared have both been deduced from the same elements, both being deductions from the same differential equation, $dy=-yap^x dx$. This equation can be made to yield, on integration, no more than one correct and complete solution of the problem proposed. I maintain that my formula, $y=g^{1-p^x}$, contains the complete solution. I also maintain that Mr. Gompertz's formula, $y=dg^{p^x}$, does *not* contain the complete solution of the problem, because it includes constants (d and a)

which had been left "indeterminate" by Mr. Gompertz in 1832, the date of publication of my formula.

In consequence of the above two constants having been left undetermined, the formula of Mr. Gompertz involves the serious error of representing the exponent of (g) to be p^x, whilst the true exponent is (p^x-1). Also, the same formula is so defective that it has never been shown to be capable of being *directly* applied to practice in the construction of a new table of mortality from newly-observed rates of mortality according to age in any population.

By means of my formula, with the constants (d and α) determined, the value of (y) may be obtained for any given age directly from newly-observed rates of mortality according to age. Mr. Gompertz, by means of his formula, containing two indeterminate constants, has never been able to do more than interpolate values of (y) between three other assumed values of (y) adopted as correct and transferred from some previously-constructed table of mortality. None of the tables so used as bases by Mr. Gompertz express any definite rate of increasing mortality according to age, nor in their construction have they been regulated by any principle of calculation superior to that commonly designated as that of the "rule of thumb."

Mr. Sprague, the advocate and representative of Mr. Gompertz, has had the opportunity of denying (if he could) the allegations which I had previously made and have now repeated, respecting the defects of Mr. Gompertz's formula, and he has omitted to make such denial. He has not denied that in Mr. Gompertz's formula the exponent of (g) is wrongly stated, nor has he denied that both (d) and (α) were left "indeterminate" by Mr. Gompertz. As to the first two defects, Mr. Sprague observes silence in his last paper. As to the third defect, he makes the allegation, that, for Mr. Gompertz's object, "it was not necessary to make any inquiry respecting the nature of the quantity (α)." This allegation does not differ much from alleging that Mr. Gompertz left (α) an indeterminate constant because he did not wish to know it more particularly.

I had previously stated, as a fact not likely to be disputed, that the object of Mr. Gompertz (as well as myself) was to find an expression for the number of living at any age (x) in terms of the mortality. Upon this Mr. Sprague remarks as follows:—"This was Mr. Edmonds's object, but "was not Mr. Gompertz's. The object of the latter was this—the mor-"tality (not the *annual* mortality) at any age (x) being proportional to "q^x (*i.e.* $=aq^x$), required a formula for the number living at the age (x). "Mr. Gompertz's formula (dg^{p^x}) gives the correct solution of his problem, "and Mr. Edmonds's formula, corrected by the introduction of L_0, gives "the correct solution of his problem." I had also remarked, that, previous to forming the differential equation $dy=-yap^xdx$, it was essential that the nature of the quantity (α) given to represent the mortality for one year should be determined. This remark, which I regarded as an axiom, has been met by a simple denial—"that it is *not* essential that the nature of (α) should be so determined." According to Mr. Sprague, it was not necessary to make any inquiry respecting the nature of the quantity (α), either before the formation of the differential equation or after the integration was completed.

It does not appear to me that these denials by Mr. Sprague can be

rendered consistent with the admission that my formula is correct for my purposes. The original quantities dealt with by Mr. Gompertz and myself are the same, and the final results arrived at by both of us are the same. There is no room for difference between the two formulæ, except in completeness—one formula being free, and the other formula *not* being free from constants which have been left "indeterminate."

It is difficult to account for Mr. Sprague's denial of the apparently obvious truth—that the two formulæ have one common object. I can attribute this denial to nothing else than misconception as to the nature of these constants (d) and (a). Mr. Sprague has fallen into numerous errors with respect to these quantities. These errors I will now proceed to correct.

I would first notice the alleged "curious inconsistency" (p. 35) of my having stated, on the same page, that two errors had been committed by Mr. Gompertz; the first error being that of using the unknown quantity (d) instead of its equivalent in known quantities, $\frac{L_0}{g}$; and the second error being that of retaining the quantity L_0 when it was manifestly superfluous. I am unable to perceive any inconsistency in this statement. The propriety of the former correction is indisputable. Also the propriety of the second correction becomes manifest when it is considered that L_0, being a common multiplier of every value of (y), can yield no information as to the proportions existing between the values of (y) at different ages, and being, therefore, superfluous, can most properly and conveniently be fully represented by unity. I now repeat the "curious inconsistency" as a truth, of which the first part has been admitted (or not denied) by Mr. Sprague, and of which the second part will never be denied by anyone besides Mr. Sprague. The two corrections indicated will convert Mr. Gompertz's formula, $y=dg^{p^x}$ into $y=\frac{1}{g}g^{p^x}=g^{p^x-1}$, which is identical with my formula—having regard to the circumstance that the (g) used by Mr. Gompertz is the reciprocal of the (g) used by me.

Respecting L_0 as distinct from unity, I have twice asserted, in opposition to Mr. Sprague, that it was superfluous and useless in the complete and correct formula of mortality for either of the three periods of human life. I now repeat the assertion, that there is no more reason for inserting L_0 into the formula of mortality than there would be for inserting any specific number of pounds sterling into a formula or table of discount of money, of which the payment is deferred (x) years. The two cases bear the closest resemblance to one another when, in the mortality table, the period of age observed is the short one intervening between "infancy" and "puberty," say from 8 to 12 years, wherein the mortality is constant and at a minimum. In the discount table, if the rate of interest be 3 per cent., each term will be to its next following term as $(1+\cdot 03)$ is to 1 : the common ratio of the series will be $(1+\cdot 03)^{-1}$, and the $(x+1)$th term will be $(1+\cdot 03)^{-x}$ when the first term is unity, say $y_0=1$.

Similarly, in the mortality table, if the mortality be constant and the finite annual decrement be $a=\cdot 0063845$, we have only to substitute this value of (a) for $\cdot 03$, in the formula above given, for the present value of £1 receivable (x) years hence, and we shall then have $y_x=(1+a)^{-x}$ to

represent the number surviving at the age (x) out of every unit of population living at age 0, or when $y_0=1$. The first term of the discount table is always taken to be $y_0=1$, and there is every reason for adopting the same course in constructing a mortality table. The quantity used to denote the finite decrement, whether in mortality tables or in discount tables, is always the difference between the ordinate 1 and the *next greater* ordinate distant by an interval of one year; but in discount tables, as in mortality tables, the finite decrement above described has always reference to the antecedent ordinate $(1+a)$, and not to the succeeding ordinate 1. Mr. Sprague expresses his opinion (p. 39) that the finite decrement in mortality tables ought to be the difference between the ordinate 1 and the *next less* ordinate. If his recommendation were adopted and extended to discount tables, it would ensue that what is now called a discount table at 3 per cent. would in future be called a discount table at 2·91262 per cent.

Mr. Sprague erroneously reasserts that my formula is defective because it does not contain the factor L_0. In contradiction I can state, that, from this formula, which does not contain either L_0 or (d), I have calculated, and afterwards published, five theoretical tables of mortality. I can also state, that, if I had used Mr. Gompertz's formula instead of my own, it would not have been necessary for my purpose to make any use of the constant (d) of that formula. This last fact supplies an excellent proof that the (d) of Mr. Gompertz's formula is superfluous and useless.

All my theoretical tables of mortality have been calculated by summation of the logarithms of the probabilities of living one year, for every year of age. The probability of living one year at any age is independent of L_0, whether my formula or Mr. Gompertz's formula be used. All my tables of mortality have been formed by commencing the summation of the logarithms of the probabilities of living one year at the age of 12 years, which is equivalent to taking unity to represent the number living at the age 12 years. By this means, in my Table A.2, the logarithms of the living at ages 12, 22 and 99, were obtained, and afterwards the numbers corresponding to these logarithms were obtained, these numbers being 1·000000, ·928658, and ·000142. In order to form Table A.1, which represents numbers living at every completed year of age, the decimal point was removed five places to the right, so that the numbers living were written thus, 100000·0, 92865·8, and 14·2. This change in the place of the decimal point was made for convenience, and not from necessity. The chief object of the change was the avoidance of the ciphers, which otherwise would have immediately followed the decimal point at ages above 80 years. The removal of the decimal point one place to the right is equivalent to, but is not really, an operation in multiplication by 10.

In the former of his two papers, Mr. Sprague, in respect of the constant (d) in Mr. Gompertz's formula, remarked (p. 291)—"It is really an essential part of the formula, and *as such is employed by Mr. Edmonds himself*." In answer, I remarked that there was not the remotest approach to truth in this statement. I added, also, that no attempt was made to adduce any proof of such an improbable statement. The attempt at proof, which was then omitted, is now adduced. This "proof" consists in alleging, that, subsequently to his erroneous statement, I have used (p. 336) the quantity (d) in order to arrive at the formula given by me as the only correct one. In this case I used the quantity in question only to eliminate

it. That is to say, I used the quantity (d) for no other purpose than to prove that it was of no use!

In my last paper I objected to one of the proposed corrections in the published process of integration of Mr. Gompertz's formula, by stating that the equation as corrected (by Mr. Gompertz or by his authority) was more defective, in form at least, than was the equation intended to be corrected. I stated that the proposed correction would leave no new constant on either side of the equation. Mr. Sprague does not deny the justness of my criticism, but he has assumed the appearance of contradicting me, by referring to the place (p. 289) of a correction *made by himself*, such correction being a confirmation of my statement. I had stated that the erroneous (b) had been abolished on one side without the correct (d) appearing, as it ought to have done, on the other side of the equation. Mr. Sprague's correction (p. 290) is, that (d) ought to have appeared in the place designated by me, but that Mr. Gompertz had introduced it at one step later in his process of integration.

This misplacement by Mr. Gompertz of the letter (d) is of high significance, as it serves to show the cause of Mr. Gompertz's failure to determine the nature of the quantity (d), and also serves to confirm a statement made by me in October, 1860 (p. 176), that the integral $y=dg^{p^x}$ was probably known to Mr. Gompertz before the corresponding differential was known to him, and that the published apparent process of integration was little more than a process of differentiation inverted.

The course of proceeding of Mr. Gompertz appears to have been the following:—after he had made the discovery, that in the period of human life extending from the age of 15 to the age of 55 years, the number living at any age (x) was represented by $y=dg^{p^x}$, the first step was to take the hyperbolic logarithms of the quantities on both sides of the equation, whence he obtained the equation—

$$\log.\ y=\log.\ d+\log.\ g\times p^x.$$

The second step was to differentiate the last equation, so as to obtain the equation—

$$\frac{dy}{y}=\log.\ g\times\log.\ p\times p^x dx.$$

The third and last step was to put

$$\log.\ g\times\log.\ p=-a,$$

the quantity (a) being an "indeterminate constant," so far as Mr. Gompertz is concerned, but having been since ascertained by me to represent the annual mortality at the age 0, or at the infinitely small interval of time next the age 0.

The published process of integration by Mr. Gompertz appears to have been derived from the above process of differentiation. The constant (d) was mistakenly supposed to be a simple quantity, easily determinable by arithmetical process, and consisting of no more than one factor, respecting which no information was required. The integration was partially performed, having been performed in the same manner as it would have been if the constant (d) had been absent, and the integral to be arrived at had been $\log.\ y=\log.\ g\times p^x$. By his erroneous first step in the process of

integration, Mr. Gompertz finds that log. y=log. $g \times p^x$; and immediately afterwards declares, without stating any reason for the addition, that log. y=log. d+log. $g \times p^x$. It is only by comparing the process of integration with the foregoing process of differentiation that the misplacement of (d) is accounted for. If the process of integration had been a genuine one, Mr. Gompertz could hardly have failed to inquire for and determine the value of the constant (d) in terms of L_0 and (g). In ordinary cases no serious error would have ensued from the course of proceeding adopted by Mr. Gompertz; but, in the present case, he had the misfortune to stumble on a constant (d), which, containing (g), the second constant of the formula, would, when known, have changed his formula from $y=dg^{p^x}$ into $y=L_0 \times g^{p^x-1}$.

I believe that I am the first person who has described the quantity (α) and used it in the construction of tables of mortality. This newly-described quantity is such that $e^\alpha=1+a$, so that α=hyp. log. of $(1+a)$. The quantity (α) represents the decrement of a unit of life, in the infinitely small interval of time next the age 0, multiplied by the infinite number of intervals into which the year is supposed to be divided, and is$=\frac{\alpha}{\nu} \times \nu=\alpha$. The quantity ($a$) is the finite decrement in one year, and measures the difference between two consecutive ordinates, separated by an interval of one year in age, when the smaller ordinate is represented by unity.

There has been an admission made by Mr. Sprague (and before noticed) which affords indirect evidence that Mr. Gompertz was *not* acquainted with the value or nature of (α). The admission referred to is, that it was not necessary for Mr. Gompertz to inquire what was the value of (α). According to my own experience, this admission is tantamount to proof that he did not know the value of the quantity in question. For, in my own case, after arriving at the knowledge of the correct formula of mortality $y=10^{\frac{k^2\alpha}{\lambda p}(1-p^x)}$, or $y=g^{1-p^x}$, I failed in my first endeavours to construct a theoretical table of mortality, through using the finite annual decrement (when the mortality was constant) to represent (α). My attention having been fixed on the subject, I soon found out the cause of my failure, and afterwards used with success the quantity α=hyp. log. $(1+a)$, instead of the quantity (a) first used by me. If Mr. Gompertz had been acquainted with the true formula of mortality $y=g^{1-p^x}$, and if he had attempted to construct a table of mortality directly from such formula (without the aid of a previously-constructed "rule of thumb" table of mortality), he would probably have succeeded, like me, in obtaining the value of (α) in terms of the finite annual decrement. But it is not alleged that Mr. Gompertz ever made such attempt, or came near the borders of the discovery made by me.

In relation to the comparative values of (α) and (a), I would call attention to what appears to me a "curious inconsistency" contained in a criticism of Mr. Sprague, at page 39. He says of me—"In calculating "the actual mortality, he supposes the rate of mortality constant throughout "the year. This is contrary to the fundamental hypothesis, that the mor- "tality at any age (x), integral or fractional, is ap^x." In reply, I would ask the critic to inform the public (if he can) how such a supposition

can be dispensed with, when the object is to express in finite terms the mortality existing during an infinitely small portion of time at a given age. I would also observe, that although the annual mortality at the precise age (x) is always expressible in exact terms by ap^x, yet this fact is perfectly consistent with the fact of the mortality being constant, as it commonly is in all tables of mortality, for a small interval of age immediately preceding the age of 15 years. At that interval of age (p) becomes equal to unity, and ap^x then becomes equal to (a).

Being the first discoverer of the quantity (a) as existing in all tables of mortality founded upon rates of mortality increasing in geometric progression, I have been obliged to describe it by some name. I have selected the name of "annual mortality," because such name, applied to a point of age, conveys nearly the same idea as is conveyed by the term annual mortality applied to an entire year of age. The annual mortality at any entire year of age is commonly represented by the numbers who have died in that year of age, divided by the average number constantly living in that year of age, during the space of one year. Such annual mortality for an entire year of age will be represented nearly by the annual mortality corresponding to the point of age marked by the middle of the year of age observed. That is to say, the annual mortality during the (x)th year and the ($x+1$)th year of age will be represented *nearly* by $ap^{x-\frac{1}{2}}$ and $ap^{x+\frac{1}{2}}$. These expressions are in accordance with my use of the term "annual mortality," to represent by ap^x the mortality experienced at the precise age (x) intermediate between the (x)th and ($x+1$)th years of age.

The recent paper of Mr. Sprague contains several errors consequent on his imperfect knowledge of the signification of the quantities (a) and (ap^x), and consequent also on his confounding together three perfectly distinct things—the annual mortality during one entire year of age, the annual mortality at the beginning of the year of age, and the probability of a person alive at the beginning dying before the end of the year of age. At page 43, Mr. Sprague says that I have erroneously stated that my law of mortality is such, that if the mortality at the age 25 were 1 per cent., and at age 45 it were 1·80 per cent., then the annual mortality at the intermediate age would be found by interpolating the terms of a geometric series; and that I have also erroneously stated that the annual rate of mortality increases in a geometric progression, of which the common ratio is (p). His words are these:—"Now, bearing in mind the distinction "already pointed out, that if ap^x be the rate of mortality at the age (x), "then the annual mortality at that age will be $1-g^{(p-1)p^x}$, it is obvious "that the annual rate of mortality does not increase in geometric progres-"sion; but, on the contrary, the chance of living a year ($=g^{(p-1)p^x}$) "decreases in a geometric progression."

In this attempt to correct an error which has no existence, Mr. Sprague has committed no less than *four* errors. The first error is that of confounding together the annual mortality at the beginning of a year of age with the probability of dying during the year. The second error is that of using the words "annual mortality at the age x" when he ought to have designated some particular year of age (either the (x)th or ($x+1$)th year), to which the probability of dying in one year is intended to be applicable. The third error is that of saying that the *chance* of living one year decreases in a geometric progression; whilst the fact is, that the *logarithm of the*

chance of living one year decreases in a geometric progression. The fourth error is a verbal one, the words "*but on the contrary*" being used when there is no contrariety, real or apparent. With regard to the first two errors committed by Mr. Sprague, I would remark, whilst using his own language, "that they were not to be expected from the veriest tyro in actuarial science." With regard to his third error, I would remark, that it was not to be expected from anyone having the slightest knowledge of Mr. Gompertz's formula or my formula of mortality.

I have now to call the particular attention of the reader to an extraordinary charge of error brought against me by Mr. Sprague. The alleged error has no existence except in the imagination of Mr. Sprague. It is only by manifest misrepresentation, first of my words, and then of my figures, that the alleged error is supported. I now give the entire passage in which the charge is made. The reader will perceive that the passage is self-contradictory in its most essential particular. At page 39, Mr. Sprague writes as follows:—

. . . . "And there will remain no doubt that he has fallen into two "very distinct and serious errors. In the first place, he has taken to "denote the annual mortality a quantity (a) such that the number living "at the commencement of the year is to the number living at its end as "$(1+a)$ is to 1. Instead of this, the correct assumption is to take (a) "such that the number entering on the year is to the number surviving it "as 1 to $1-a$. This is a fundamental error, and one that was not to be "expected from the veriest tyro in actuarial science. To put the point in "another way:—the annual mortality at any age (x) is found by com-"paring the finite decrement for one year with the number living at the "*beginning* of the year, whereas Mr. Edmonds has compared it with the "number living at the *end* of the year! The practical result of this error "will be, that supposing (α) to be the mortality at the beginning of a "year, Mr. Edmonds makes the actual mortality of the year greater than "(α), instead of less, as it obviously should be. To take the figures given "by him on page 335, if $\alpha=\cdot0063643$, then will the actual mortality of "the year be $1-e^{-\alpha}$ or $\cdot0063440$, instead of $\cdot0063845$ as stated."

With regard to the above extraordinary charge, I would first observe that Mr. Sprague is in error if he believes that the finite decrement for one year when the mortality is constant, in respect of a unit of life at the beginning of the year, as given by me, differs in any respect, except that of form, from the similar finite decrement given by himself. The finite decrement given by me is (a) in respect of $(1+a)$ living at the beginning of the year, and is consequently $\frac{a}{1+a}$ in respect of a unit of life at the beginning of the year. The finite decrement given by Mr. Sprague in respect of a unit of life is $1-e^{-\alpha}=1-\frac{1}{e^{\alpha}}=1-\frac{1}{1+a}=\frac{a}{1+a}$, a quantity which is identical with my own.

I have next to observe that Mr. Sprague has said what is *contrary to the truth* when he states (in contradiction to what he himself had previously stated in the same paragraph) that "Mr. Edmonds has compared the finite decrement for one year with the number living at the *end* of the year." The statement made by me (p. 335), and misrepresented by Mr. Sprague, is the following:—"(a) is the finite decrement for one year, and is such that

"the number living at the beginning of the year is to the number surviving "at the end of the same year as $(1+a)$ is to 1." It appears to me impossible for anyone to fail to perceive from the sentence quoted, that (a) was given by me as the decrement in one year out of $(1+a)$ persons alive at the beginning of the year. Indeed, Mr. Sprague shows that there was no misunderstanding of my words on his part, for he writes thus at foot of page 38:—"Mr. Edmonds supposes the annual mortality to be such "that out of $(1+a)$ persons alive at the beginning of the year, (a) die in "the course of the year, so that the annual mortality is $\frac{a}{1+a}$."

Mr. Sprague having misrepresented my words, proceeds, in the next place, to misrepresent my figures; and having obtained a correct result by his own method, compares it with an imaginary result, which, without any foundation, he alleges to have been given by me. The words used by him, "instead of ·0063845 as stated," are *not true*—no such statement having been made by me. The finite decrement in one year, in respect of a unit of life at the beginning of the year, has been stated by me to be $\frac{a}{1+a}$. On substitution of the value given for (a), we obtain for the value of such decrement $\frac{\cdot0063845}{1\cdot0063845}$, which is equal to ·0063440, being the same result as that obtained by Mr. Sprague. This misrepresentation of my figures is the less excusable from the circumstance that Mr. Sprague had before him and was quoting from my published Tables A.1 and A.2. On inspection of these Tables (between ages 8 and 12 years), he might easily have seen that the results, given by him as new, and corrections of my results, are identical with results published by me thirty years ago.

I have been blamed by Mr. Sprague for using the words "homely" and "transcendental" in describing the substance of the law or part of the law of human mortality discovered by Mr. Gompertz. I could find no more suitable words to represent the facts which I wished to communicate. The facts themselves I regarded as of the highest importance in the question of a disputed claim between Mr. Gompertz and myself. I believe that if the facts communicated in the paragraph complained of had been sooner known to Mr. De Morgan and to the public, no charge of "unfair suppression" would ever have been made against me. In that paragraph I stated that the universal law of human mortality was capable of being correctly and precisely expressed by three geometrical series—the common term of each of which being ap^x, and the common ratio (p) having a fixed and determined value in each of the three different periods of human life. I also therein stated that the formula $y=g^{1-p^x}$ was a mathematical consequence of the above-described simple law of mortality, but that the knowledge of this formula did not necessarily either precede or follow the knowledge of the true and exact law of human mortality signified by the expression ap^x.

In the letter from Mr. Gompertz there occurs (p. 297) the following passage, in which my claim is incorrectly described:—

"I wish, in this letter, to notify my thanks to Mr. Sprague, whom I "have not the honour of personally knowing, for his kind and able paper "in vindication of my claim to be the sole independent publisher of a

"theorem which, I am gratified to say, appears of importance to the "scientific world. I say the sole independent publisher, though the fact "of my being the first independent discoverer has not been denied me even "by the gentleman who claims to be the second discoverer, because that "claim, should it be ever proved to be a just one, would not interfere "with me."

In the above passage the question at issue between Mr. Gompertz and myself has been misrepresented. The claim described as mine by Mr. Gompertz is not the claim which I have made. I have not yet made any claim to be either the first or second discoverer of the theorem alluded to—comprised in the formula $y=dg^{p^x}$, and supposed to express correctly and completely the relation at every age between the mortality and the number living. I acknowledged, in my *Life Tables*, published in 1832, that Mr. Gompertz had preceded me, in using, for a similar purpose, a formula greatly resembling my own formula. I did not allude to the defects in Mr. Gompertz's formula, which I then believed to exist, but which I have not had the opportunity of proving to exist until the present moment. As to putting in a claim to be "*second*" discoverer, either of the law of human mortality or of the theorem alluded to, the idea never entered my mind of doing anything so futile.

The claim made by me in 1832 (adopting the phraseology of Mr. Gompertz) amounted to this—that I was the sole independent publisher and first discoverer of the universal law of human mortality as applicable to three distinct periods of human life—*before*, *during*, and *after* maturity; that the mortality at any age (x) in each period was represented by ap^x; and that the values of (p) for the three several periods of life in all populations were $\frac{1}{1\cdot48}$, 1·03, and 1·08 respectively. This discovery was complete before I had any knowledge of the existence of any connexion between such discovery and the theorem of Mr. Gompertz.

Until the present time I have not made any claim which affects Mr. Gompertz's title to the credit of discovering the formula which has been supposed to exhibit, "*for a long period of man's life*," the true and exact relation between the mortality and the numbers living at different ages. Now, however, after having proved the existence of two serious defects in his formula, I am about to make a claim which will affect the value of Mr. Gompertz's claim. The claim which I am about to make is founded on the belief which I entertain that the formula of Mr. Gompertz ceased to be of any value to the public in the year 1832, the date of publication of my formula.

Now, in 1862, I lay claim to the credit of being the first discoverer of the only existing formula which exhibits the true and exact relation between the mortality and the numbers living according to age, in either of the three periods of human life. I have produced abundant evidence of the fact, that the formula of Mr. Gompertz contains two constants which were left "indeterminate" by Mr. Gompertz, but have been "determined" by me. This evidence has been produced before an accepted advocate and representative of Mr. Gompertz. The truth of my allegations respecting the two constants has not been denied, and the representative of Mr. Gompertz has shown himself unable to prove that either of these constants had ever been "determined" by Mr. Gompertz.

Report made by Mr. Samuel Brown to the International Statistical Congress, as to the Institute of Actuaries, &c.

The mere collection of statistics would be of little service to the public unless they were carefully analysed and compared, and those practical results deduced from them which may tend to benefit society at large. In every branch of statistics there are ardent inquirers into truth, ready to examine the facts collected, with the view of deducing therefrom those laws which seem to govern the mental, the moral, and the social as well as the material world. But there is one class of facts in which, in this country more especially, the results of statistical labours have been turned to the most practical use for the public advantage, and have led to the establishment of Companies of the highest importance, both in a social and commercial point of view. The questions regarding population, the causes of its increase or decrease, marriages, births and deaths, the diseases which end in the latter, and seasons of scarcity or plenty, which affect all alike, engaged at an early period the attention of English as well as foreign statists. Dr. Farr, in his Report on the Programme of this Congress, has ably pointed out how the mathematical school of statistics, commencing with Halley, and the construction of his Life Table, drew attention to the important application of population statistics to estimate the value of human life. There can be no doubt that it was owing to the scientific character of the men who first studied the question of vital statistics that life assurance in this country has been developed to so extraordinary an extent, and is established on so firm a basis. Halley, Demoivre, and Simpson laid the foundations of the science. Dr. Price exposed, with great ability, the false schemes of numerous Annuity Societies, and the ignorance with which they were conducted, and was directly concerned in the establishment, on a sound basis, of the first Society which put to practical use the science of life statistics—the Equitable Assurance Society. The appointment of his nephew, Mr. Morgan, as the Actuary of the Society, ensured its success, for it required both prudence and caution, and high mathematical attainments, to introduce principles then so novel and distrusted. Since then, the system has spread throughout the kingdom, till, by means of various Public Companies and Mutual Associations, a sum considerably exceeding £200,000,000 is secured for the families or representatives of the assured in this country, and which requires at least £6,000,000 in annual payments during the lives of the assured to meet these large accumulations at their deaths. Nor is this the only practical application of these doctrines of life statistics. Reversionary Interest Societies, for purchasing life interests or reversions depending on human life; Annuity Societies, to secure provision for old age or for survivorship in marriage; Endowment Societies, for providing capital to educate or put forward children in life; Friendly Societies and Sick Funds, for affording relief in sickness; all flourish or decay in proportion as the managers of them either understand the science of the probabilities of human life, or consult those who do; or, on the other hand, proceed in ignorance of, or persist in a wilful indifference to, the true laws which govern these events. All these interests, so great and varied, engage the attention of a body of men who, as actuaries of Assurance Companies, are bound to study the scientific principles on which the safety and prosperity of these Companies depend, and by their public character and mathematical education naturally take an interest in all those statistical

questions, in the investigation of which the doctrines of probability can be applied either to discover their laws of operation or to educe useful results.

With the view of combining the knowledge and stimulating the energy of the men engaged in these intellectual pursuits and useful labours, and, still more, of training the younger members of the profession to a more ardent cultivation of the science, the Institute of Actuaries was constituted in January, 1849, and the late John Finlaison, Esq., the Government Actuary, was elected the first President. On the recent and lamented death of Mr. Finlaison, Charles Jellicoe, Esq., was unanimously elected his successor.

The Institute was declared to be founded for the purpose of elevating the attainments and status and promoting the general efficiency of all who are engaged in occupations connected with the pursuits of an actuary, and for the extension and improvement of the data and methods of the science which has its origin in the application of the doctrine of probabilities to the affairs of life, and from which life assurance, annuity, reversionary interest, and other analagous institutions derive their principles of operation. It embraces, as its peculiar province of inquiry, all monetary questions involving a consideration of the separate or combined effects of interest and probability.

It will thus be seen that, under one or other of these heads, it comprises the study of most of the great questions of political or social economy, from the statistics of which the laws of their occurrence or recurrence can be deduced by the doctrines of probability, even though they may not give rise to public Societies to carry them into actual practice.

The Institute consists of Contributing Members, Fellows and Associates, and Non-contributing (namely, Honorary, Foreign and Corresponding) Members. At the present time the number of Fellows and Official Associates is about 66, of the Associates about 80.

Amongst the Honorary Members are enrolled the names of the late Mr. G. R. Porter, of the Board of Trade, whose statistical labours are so well known; Mr. Charles Babbage, F.R.S.; Professor De Morgan; Dr. Farr, F.R.S.; Mr. Benjamin Gompertz, F.R.S.; Mr. Peter Gray; Sir J. W. Lubbock, Bart., F.R.S.; the Right Hon. Lord Overstone, F.R.S.; Mr. John Tidd Pratt, M. Quételet, and Professor Sylvester, F.R.S. The Corresponding Members in Belgium, France, Germany, Holland, Russia, Sweden, and the United States of America, are connected mostly with the great assurance institutions of those countries, and have, many of them, rendered, by their writings or contributions to the *Journal*, valuable service to the literature of the subject. Amongst these may be mentioned MM. Joliat, Lafond, Le Hir, Maas, Pouget, Bruggemann, Hopf (of Gotha), Lazarus (of Hamburg), Masius, Daninos, Dr. Curtis, Professor Silliman, and Mr. Sheppard Homans.

The Institute is governed by a President and Council. The Council consists of Fellows and Official Associates, out of whom the Members elect by ballot, annually, a President, four Vice-Presidents, a Treasurer, and two Secretaries.

One of the most efficient means of carrying out the purposes of the Institute has been the establishment of a system of voluntary examinations; the examiners being Members of the Institute, and a moiety at least of them being Fellows. The examination comprises mathematical theory, vital statistics, computation and construction of tables, bookkeeping, and office

routine. It consists of three parts, separated from each other by at least one year; and no Member is eligible for the final examination until he has attained the full age of 21 years. The first year's examination is confined principally to questions in general mathematics and geometry. The second year's examination comprises more difficult questions in the theory of logarithms, of probabilities, tables of mortality and annuities, and assurances on lives and survivorships. The third year's examination extends to (1) *Life Assurance Finance*—the construction and graduation of tables of mortality, the methods of determining surplus and distributing it in an Assurance Society; (2) *Legal Principles*—Acts of Parliament, partnerships, probates and letters of administration, assignments, bankruptcies; (3) *Statistics*—methods of collecting and arranging data, tests of accuracy, general system of the country's finance, funded and unfunded debt, and fiscal arrangements, taxation; (4) *Currency, Banking and Investments*—currency (metallic and paper), nature of banking, bills of exchange, comparative value of securities, high and low prices, fluctuations in the rate of interest; and (5) *Miscellaneous*—bookkeeping, auditing, valuation of marketable securities, and approximate calculations. Every year since the establishment of the Institute a number of candidates have presented themselves for these voluntary examinations, and several, who have passed, have distinguished themselves by the positions they have taken in life, and by their writings and contributions to the *Journal* of the Institute.

The *Assurance Magazine and Journal of the Institute of Actuaries* was commenced soon after the establishment of the Institute itself. It has now reached the ninth volume, and comprises a variety of articles of the greatest importance and public interest. The contributors to it are mostly, though not exclusively, members of the Institute. Besides original papers and those which are read at the sessional meetings, it contains foreign intelligence, relating to the vital statistics, Assurance Companies (whether for life, fire, or any other branch), Friendly Societies, &c., of all foreign countries; reviews of new works, reports of Companies, correspondence, and the proceedings of the Institute. It would be impossible, in the space allotted to me now, to give any idea of the many valuable original articles, either read at the sessional meetings of the Institute, or prepared expressly for the *Journal*, which are to be found in these nine volumes. Amongst the contributors of some of the most important papers, are Messrs. Jellicoe, Hardy, Hendriks, Higham, Gray, Hopf, Farren, Pinckard, Bunyon, Tucker, Lazarus, Hodge, Day, Porter, &c. Professor De Morgan has also contributed some original arithmetical papers of great value.

At the sessional meetings of the Institute, which are held monthly from November to May, some very interesting papers have been read and discussions held thereon. On all questions of great public interest it has been usual to invite gentlemen, not members of the Institute, who have either written on, or are distinguished for their knowledge of, the subject to be considered; and thus, by the opposition of views, not merely to elicit a lively debate, but to obtain, what should be the result of all discussion, the nearest possible approach to the truth. Amongst the questions of great public interest which have thus been debated, may be mentioned—"The injustice of taxing temporary annuities at the same rate as perpetual annuities." "The rates of mortality, as deduced from the actual experience of Assurance Companies, such as the Eagle, the Economic, the Life Assurance Bank of Gotha, the invalid lives in the Clerical, Medical and General;

risks of climate in America, in the Mutual Life of New York." "The sickness and mortality in Friendly Societies in Great Britain, France and Germany." "Decimal coinage." "The early history of life assurance and annuities, and the recovery by Mr. Hendriks of the lost treatise of the Grand Pensionary, De Witt, on the subject of life annuities." "The progress of the calculus of probabilities." "The true methods of valuing and the marketable values of life interests and reversionary properties." "On the rates of interest for the use of money in ancient and modern times." "On the annual reports of the Registrar-General, and how far the inordinate mortality of the country is controllable by human agency." These, and many other mathematical and professional questions, which, in connexion with life statistics, indirectly but deeply affect the public interest, show how many and varied are the questions opened up for discussion.

The Institute of Actuaries, having so many subjects in common with the Statistical Society, naturally takes the warmest interest in its labours and prosperity. It has for several years worked in harmony, and used the same apartments, and many of its members are also members of the Statistical Society, joining in the debates, aiding the operations, and contributing to the Journal of the latter Society.

It would appear, then, that, by their mental education and practical experience, the profession of actuaries is peculiarly called upon to take a warm interest in the great statistical subjects which have been examined and discussed at this and the previous Congresses. They cannot but feel it their duty and their pleasure to aid in improving the methods of collecting facts and forming statistical tables, in comparing their results in this and other countries, in analysing and correcting conflicting statements which are frequently given as facts, and in endeavouring to deduce from the vast collections which have been already made, or may hereafter be formed by the various Governments represented at this important Congress, those laws which affect the moral, the social, or the physical well-being of all classes of the community. They are ardently anxious to co-operate in a work of such great public advantage, and to assist, by every means in their power, to render practically useful the knowledge which is thus gleaned in the wide field of statistics.

CORRESPONDENCE.

ON THE SIGNIFICANCE OF THE EXPRESSION $\frac{1}{1+a}-(1-v)$.

To the Editor of the Assurance Magazine.

Sir,—It is a trite remark, that we know, or think we know, many things, while we may be very far from apprehending their full significance. I have just met with an instance of this, which appears to me to be not devoid of interest.

We have all been long familiar with the expression, first deduced by Mr. Milne, for the annual premium for assurance—namely, $\frac{1}{1+a}-(1-v)$. This, when attention is called to it, must be admitted to be a remarkable

expression; and yet there has not, so far as I know, any attempt been made to interpret it. Circumstances have led me to attend to it; and I now, in consequence, propose the following

THEOREM.

If a' be the present value of an annuity due of £1, payable during the continuance of any status, then will the annual premium, π, for a benefit of £1, receivable at the end of the year in which that status fails, be determinable by the equation—

$$\pi = \frac{1}{a'} - (1-v).$$

For, if the arrangement were that the £1 benefit should be receivable *now*, instead of at the end of the year in which the status fails, the equitable premium would obviously be $\frac{1}{a'}$, that is, the annuity due which that £1 would purchase; but inasmuch as the £1 is not receivable now, each payment of the above premium made while the £1 is withheld will have to be diminished by the *discount* (not the *interest*) on £1. Hence the foregoing equation holds.

Attention to the terms of the proposition will, of course, enable us to distinguish the cases in which it is applicable from those in which it is not.

1. It is applicable to assurances on single lives, and on joint lives, of whatsoever number the combination may consist, when the assurance is, in the one case for the whole of life, and in the other for the joint duration. This is well known; but it is not applicable in either of these cases if the assurance is only for a term, inasmuch as the status will not *necessarily* fail, and the benefit consequently will not *necessarily* become payable, during or at the end of the term.

2. It is applicable to an assurance and endowment on either a single life or a combination of any number of joint lives; for here the benefit will necessarily become payable, either during the term, by the failure of the status, or at its termination if the status continue to subsist.

To test this:—We know that the annual premium for this benefit is

$$\frac{M_x - M_{x+n} - D_x}{N_{x-1} - N_{x+n-1}},$$

which, if we substitute in it, for M_x and M_{x+n}, their values in terms of D and N, by the formula $M_x = D_x - (1-v)N_{x-1}$, reduces to

$$\frac{D_x}{N_{x-1} - N_{x+n-1}} - (1-v).$$

And this is of the desiderated form, for the first term is the annuity due, for n years, on (x), that £1 paid now will purchase.

This formula was given (first, I believe) by your correspondents Mr. Sprague and "H. A. S.," on pp. 112 and 117 of your eighth volume; and it was consideration of it, with its analogy to the whole life premium for assurance, that led me into the present inquiry.

3. The property enunciated is applicable to assurances on last survivors if the premium is payable till the benefit becomes due, but not otherwise.

4. It is not applicable to survivorship assurances, for here the benefit does not *necessarily* become due by the failure of the status.

5. It is applicable to the case of a sum payable at the end of a term, and subject to no contingency.

Thus, the sum payable in n years being £1, its present value is v^n. And the present value of an annuity of £1 for n years being $\frac{1-v^n}{r}$, anticipating all the payments a year, we have for that of the same annuity due, $\frac{1-v^n}{vr}$, or $\frac{1-v^n}{1-v}$. Now let π be the premium required—

$$\text{Hence } \frac{\pi(1-v^n)}{1-v}=v^n, \text{ and } \therefore \pi=\frac{(1-v)v^n}{1-v^n}.$$

Adding and subtracting $(1-v)$, we get—

$$\frac{(1-v)v^n}{1-v^n}+(1-v)-(1-v)=\frac{(1-v)v^n+(1-v)(1-v^n)}{1-v^n}-(1-v)$$

$$=\frac{1-v}{1-v^n}-(1-v),$$

which verifies the property, since $\frac{1-v}{1-v^n}$ is, as above, the annuity due of n payments that £1 will purchase.

I am, Sir,

Your most obedient servant,

7, *St. Paul's Villas, Camden Town,* P. GRAY.
Dec. 10*th*, 1861.

P.S.—Since the foregoing was written, the following additional application of the principle employed has occurred to me:—

THEOREM.

If a' denote the present value of an annuity due of £1, on the continuance of a specified status, then will the present value of £1, receivable at the end of the year in which that status shall fail, be denoted by

$$1-(1-v)a'.$$

£1, payable now, is worth, of course, £1. To defer its payment one year, the equitable consideration would be, the disbursement by the debtor of $1-v$; and, obviously, the deferring of the payment to the failure of the status would be compensated by the disbursement of $1-v$ at the commencement of each year during the subsistence of the status. Hence, the present value of all these disbursements being $(1-v)a'$, that of £1, payable at the end of the year in which the status shall fail, is

$$1-(1-v)a'.$$

We know the truth of this in regard to contingent assurances. But it is also true in the case in which no contingency is involved, that is, when the status is simply a term of years, n. For, in this case, $a'=\frac{1-v^n}{1-v}$. Hence, by substitution—

$$1-(1-v)a'=(1-v)\frac{1-v^n}{1-v}=1-(1-v^n)=v^n.$$

P. G.

ON A METHOD OF ESTIMATING THE INCREASE OF RATE PUT ON ENDOWMENT ASSURANCES TO MEET DETERIORATION.

To the Editor of the Assurance Magazine.

SIR,—Tables of premiums for assurances payable at specific ages (50, 60, &c.), or at death, if before, are now contained in the prospectuses of most Companies, and I doubt not the transactions arising from them are proportionally numerous. In the use of such tables it will often happen that, from the arbitrary increase of rate, founded on adverse medical reports or other causes, it is not obvious how many years are thereby assumed to be added to the age of the assured; and I would submit to you the following simple method of arriving at this by inspection.

Since, for an assurance and endowment payable in n years, to be paid for in n annual premiums,

$$\pi_x = \frac{1}{1 + a_{x\,\overline{n-1|}}} + v - 1,$$

then

$$\frac{1}{1 + \pi_x - v} = 1 + a_{x\,\overline{n-1|}};$$

and, from the form in which Mr. Sang has tabulated his Short Annuity Values, we have but to substitute, in the last equation, for π_x, the increased premium per £1 (diminished by its proportion for Office commission), and look for the value in the "short annuity" column of the nth page of Sang—the age corresponding thereto, or to the annuity value nearest to it, being the advanced age wanted.

For an example, take a case of an Assurance Office which charges the Carlisle premium at 3 per cent., increased by 10 per cent. for management; the benefit, an assurance and endowment payable at death or on the assured (presently aged 27) attaining the age of 60. Therefore, $n=33$; and suppose that 10*s*. has been added to the Office premium per £100, π, the net premium per £1, is, therefore,

$$\frac{1}{110}(£2.\ 14s.\ 11d. + 10s.) = \cdot 0295075,$$

and

$$\frac{1}{1 + \pi - v} = 17\cdot 055.$$

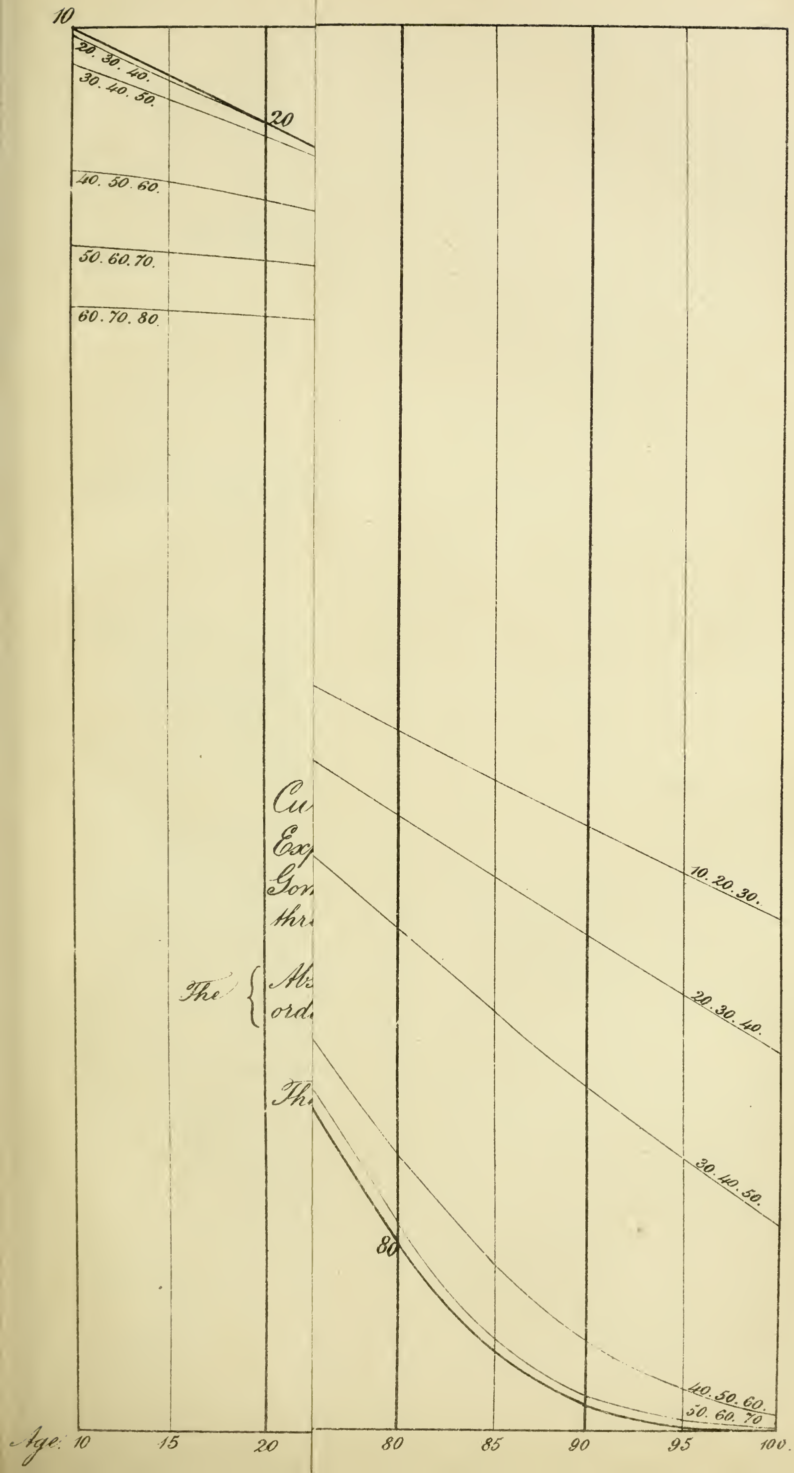
On the thirty-third page of *Sang's Tables*, we find the nearest "short annuity" value to be £17·097, being the amount opposite age 40; and the premium charged by the Office is, therefore, slightly over the premium for assurance and endowment payable in 33 years on a life of the age of 40.

I am, Sir,

Your most obedient servant,

Aberdeen, 16th December, 1861. H. AMBROSE SMITH.

P.S.—A wish to compare the results of the approximation for the values of annuities on three joint lives, given in your last Number, with the table in *Jones*, vol. ii., p. 1087, has led me to the discovery that the latter is based on the rate of *five* per cent., and not on *three* per cent., as stated. Perhaps it may be worth your making this known, for the benefit of others desirous of putting the table to a like use. I have re-computed it, and find it correct. H. A. S.



10
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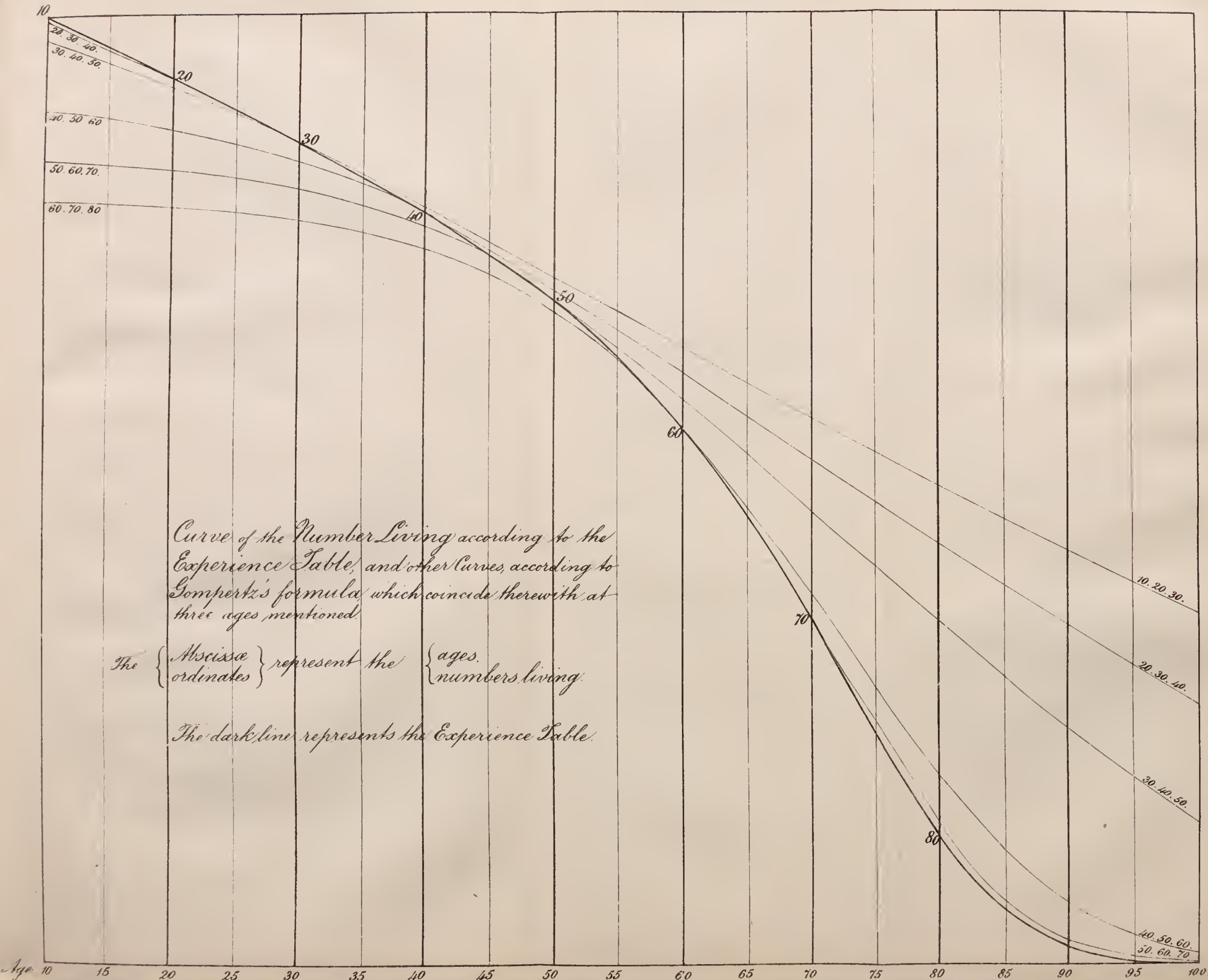
Curve of the Number Living according to the Experience Table, and other Curves, according to Gompertz's formula which coincide therewith at three ages mentioned.
The {Abscissæ / ordinates} represent the {ages. / numbers living.
The dark line represents the Experience Table.
10.20.30.
20.30.40.
30.40.50.
40.50.60
50.60.70.
60.70.80
Age
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35
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45
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55
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85
90
95
100.

THE

ASSURANCE MAGAZINE,

AND

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

Observations on Gompertz's Law of Mortality and the Dependence between it and Simpson's Rule for finding the Value of an Annuity on Three Lives. By W. S. B. Woolhouse, F.R.A.S., F.I.A., F.S.S., *&c.*

[Read before the Institute, 30th December, 1861, and printed by order of the Council.]

In the correspondence that has recently taken place respecting the origin of Gompertz's formula expressive of the law of mortality, it has been stated, much to my surprise, that the investigation first given by that gentleman in the *Philosophical Transactions of the Royal Society* for the year 1825 was defective and inaccurate. I do not here propose to reopen the controversy on the subject of the originality of discovery, or to enter upon a renewed discussion of certain other allegations that have been so efficiently disposed of by Professor De Morgan and Mr. Sprague; but, after the statement to which I have more particularly alluded, it is only right that I should avail myself of the present opportunity to communicate the circumstance that Mr. Gompertz kindly presented me with a copy of his valuable Memoir in January, 1839,* in which the various misprints had, in the most legible manner, been previously corrected by himself, and that the processes so corrected are perfectly accurate in every respect. In order that any one may be

* Curiously enough, I wrote the date in the volume at the time of receiving it.

enabled to judge of this for himself, I shall here give a faithful transcription of that portion of the Memoir which contains the mathematical investigation of the formula, as it will occupy but little space.

"If L_x be the number living at the age x; we shall have $aL_x \times q^x \dot{x}$ for the fluxion of the number of deaths $= -(L_x)^{\cdot}$; $\therefore a\dot{x}q^x = -\frac{\dot{L}_x}{L_x}$, $\therefore aq^x = -$hyp. log $q \times$ hyp. log $\frac{L_x}{d}$, and putting the common logarithm of $\frac{1}{q} \times$ square of the hyperbolic logarithm of $10 = \frac{a}{c}$, we have $c.q^x =$ common logarithm of $\frac{L_x}{d}$; d being a constant quantity, and therefore L_x or the number of persons living at the age $x = d(g)^{q^x}$; g being put for the number whose common logarithm is c."

Here there is not the slightest inaccuracy, and the investigation and resulting formula are as complete as could be desired. But in stating thus much, I should, at the same time, carefully bear in mind, that, to many members of the Institute, such confirmatory evidence will not be needed, as it would be preposterous to suppose that Mr. Gompertz, one of the greatest mathematicians of Europe, could in reality bungle in working out the integration of the most elementary form of exponential. The correctness of the final result is indeed a sufficient indication that the intermediate operations, which presented no difficulty, had, by so experienced a hand, been accurately performed.

Mr. Gompertz, who possesses the conscientiousness of a great mind, did not announce his formula as an absolute discovery of the true law of mortality, but, by the computation of various examples from recognised tables, he clearly established for it the important merit of representing, with a remarkable degree of accuracy, the tabular mortality throughout long periods of years; and, for the attainment of this desirable object, it is not likely to be superseded by any other theory equally free from complication.

As the letter d is commonly used as the prefix of a differential, let k be put in place of it, and the formula expressing the number of persons surviving at any age x will be

$$L_x = k(g)^{q^x}.$$

Passing into common logarithms and differences, at uniform intervals of t years, it gives

$$\log L_x = \log k + q^x \log g$$
$$\Delta \log L_x = (q^x - q^{x+t}) \log g = q^x(1-q^t) \log g$$
$$\log (\Delta \log L_x) = x \log q + \log (1-q^t) + \log^2 g$$
$$\Delta \log (\Delta \log L_x) \text{ or } (\Delta \log)^2 L_x = t \log q.$$

Therefore, if the logarithms of the numbers living at equal intervals of age be differenced, the differences will constitute a series in geometrical progression;* and if the logarithms of these last be taken and differenced, the differences so obtained will be constant.

For practical use, these relations are most convenient when exhibited according to the following table:—

Age.	(1) = log L.	(2) = Δ(1).	(3) = log (2).	Δ(3).
x $x+t$ $x+2t$ &c.	$\log k + q^x \log g$ $\log k + q^{x+t} \log g$ $\log k + q^{x+2t} \log g$ &c.	$q^x(1-q^t) \log g$ $q^{x+t}(1-q^t) \log g$ &c.	$x \log q + \log (1-q^t) + \log^2 g$ $(x+t) \log q + \log (1-q^t) + \log^2 g$ &c.	$t \log q$ &c.

If it should be required to construct a table from given values of the constants k, g, q, it will only be requisite to form the columns (1), (2), (3), in a retrogade order, the numbers in column (3) having the constant difference $t \log q$.

As the formula contains three arbitrary constants, it will follow that if the calculated mortality be laid down in a curve, it may be made to pass through any three assigned points, and that the assumption of three such points will be sufficient to determine the values of the constants. But in a curve of mortality that does not conform with Gompertz's law, the computed values of the constants will vary with the position in which the three points are taken; and if the points be assumed to be indefinitely near to each other, the curves will osculate in their vicinity. Thus, in general, any curve of mortality may be considered as accurately represented by Gompertz's formula, if the symbols k, g, q, instead of being constants, are supposed to denote functions of the age x, and to appear as variable parameters that usually undergo but gradual changes in value. As a practical example of these changes, I have calculated the following table from the numbers living according to the Experience of Seventeen Life Offices:—

* Mr. Sprague, in the October Number of the *Journal*, p. 43, states that "the chance of living a year $(=g^{(p-1)p^x})$ decreases in geometrical progression." This is either an inadvertence or a misprint, into which it is immaterial to inquire, as there can be no doubt that the logarithm of the chance of living a year was meant to be expressed. The formula given in the parenthesis is a superposed exponential.

Calculation from Experience Table, (t=10).

Age x.	$\log L = \log k + q^x \log g$.	$\Delta \log L = q^x(1-q^t)\log g$.	$\log. = \log(1-q^t) + \log^2 g + x \log q$.	$\Delta = t \log q$.	$\log k$.	$\log g$.	$\log q$.
10	5·0000000	0·0302673	8·4809737				
20	4·9697327	0·0337622	8·5284308	0·0474571	5·26212	−0·23498	0·00475
30	4·9359705	0·0402552	8·6048220	0·0763912	5·14528	0·12349	0·00764
40	4·8957153	0·0536243	8·7293616	0·1245396	5·05718	0·05128	0·01245
50	4·8420910	0·0941124	8·9736468	0·2442852	4·96673	0·00749	0·02443
60	4·7479786	0·1936470	9·2870108	0·3133640	4·93108	0·00241	0·03134
70	4·5543316	0·4308066	9·6342824	0·3472716	4·90610	0·00130	0·03473
80	4·1235250	1·0032802	0·0014222	0·3671398	4·57479	−0·00006	0·03671
90	3·1202448						

In this table it will be perceived that when Gompertz's curve is made to pass through the points indicated by the ages 10, 20, 30, the constants will be $\log k = 5·26212$, $\log g = -0·23498 = 9·76502$, $\log q = 0·00475$; when it is made to pass through the points indicated by the ages 20, 30, 40, the constants are $\log k = 5·14528$, $\log g = -0·12349$, $\log q = 0·00764$, &c. &c.; and in each case the values may be considered as appertaining to the middle age.

Mr. Edmonds, adopting Gompertz's formula as a basis of construction, has, with considerable skill and labour, succeeded in accommodating it to the usual tables of mortality by dividing human life into three periods—viz., infancy, manhood, and old age, over each of which he calculates separately from distinct data. His curve, which is partly theoretical and partly empirical, has, therefore, as it were, two intermediate stations, from which it proceeds on new branches depending on new constants. This abruptness of transition is certainly objectionable; but, although the marked discontinuity at these points does no little violence to abstract notions of mathematical accuracy and consistency, it must be conceded that the results are sufficiently near to the truth for ordinary business purposes.

I shall conclude my observations on this part of the subject by an example in full of the calculation of the curve represented by Gompertz's formula when it is made to pass through three given points; and it will be seen that this can readily be accomplished without the labour of computing the values of the constants that enter into the equation of the curve. The best method of interpolating a table of mortality by means of Gompertz's formula will also be thus indicated.

Let the given points correspond to the ages 20, 30, 40, of the

Experience Table; and let it be required to make the calculation for intervals of five years.

Taking the logarithms of the numbers living at the ages 20, 30, 40, for the first column, and following the precepts at the head of the table, as before described, we get—

Age.	(1) = log L.	(2) = Difference.	(3) = log′(2).	Diff. = t log q.
20 30 40	4·9697327 4·9359705 4·8957153	0·0337622 0·0402552	8·5284308 8·6048220	0·0763912

Here $t=10$; and, since the first line of column (3) represents the value of the expression $x \log q + \log(1-q^t) + \log^2 g$, it is evident that, to adapt the numbers to the required interval $t'=5$, we need only to replace the term $\log(1-q^t)$ by $\log(1-q^{t'})$, which will be most readily effected by applying the correction—

$$\log(1-q^{t'}) - \log(1-q^t),$$

and this is done as follows:—

$t \log q$	0·0763912	$\therefore\ t' \log q$	0·0381956
q^t	1·1923160	$q^{t'}$	1·0919320
$1-q^t$	$-$0·1923160	$1-q^{t'}$	$-$0·0919320
		$\log(1-q^{t'})$	8·9634667
		$\log(1-q^t)$	9·2840155
		Correction	9·6794512
		(3) ...	8·5284308
		(3)′ ...	8·2078820

This last result will now occupy the line of column (3) that comes immediately after age 20. The subsequent numbers in that column are obtained by successively adding the constant difference $t' \log q = 0·0381956$, as above, and those which precede are found by the subtraction of the same constant difference. After completing column (3) in this manner, and inserting the given values of log L for the ages 20, 30, 40, the latter of which will afterwards serve as checks to the accuracy of what has already been done, the required table worked, or filled in, to the left, will finally appear thus:—

Gompertz's Curve through Ages 20, 30, 40.

Age.	log L.	Δ.	(3) = log Δ.
10	4·9980491	0·0135360	8·1314908
15	4·9845131	0·0147804	8·1696864
20	4·9697327	0·0161392	8·2078820
25	4·9535935	0·0176229	8·2460776
30	4·9359705	0·0192430	8·2842732
35	4·9167275	0·0210121	8·3224688
40	4·8957153	0·0229438	8·3606644
45	4·8727715	0·0250530	8·3988600
50	4·8477185	0·0273562	8·4370556
55	4·8203623	0·0298711	8·4752512
60	4·7904912	0·0326172	8·5134468
65	4·7578740	0·0356158	8·5516424
70	4·7222582	0·0388900	8·5898380
75	4·6833682	0·0424652	8·6280336
80	4·6409030	0·0463692	8·6662292
85	4·5945338	0·0506320	8·7044248
90	4·5439018	0·0552867	8·7426204
95	4·4886151	0·0603693	8·7808160
100	4·4282458		

In this way I have made calculations for other points in the curve, and, to show their characters and peculiarities, the whole of the results are briefly comprised in the following tables :—

Logarithm of the Number Living.

Age.	Gompertz's Curve passing through Three Ages.					
	10, 20, 30.	20, 30, 40.	30, 40, 50.	40, 50, 60.	50, 60, 70.	60, 70, 80.
10	5·0000000	4·9980492	4·9888748	4·9535995	4·9261123	4·9031950
15	4·9852797	4·9845132	4·9783440	4·9493325	4·9239556	4·9017688
20	4·9697327	4·9697327	4·9661896	4·9436796	4·9208619	4·8996415
25	4·9533127	4·9535935	4·9521614	4·9361908	4·9164242	4·8964686
30	4·9359705	4·9359705	4·9359705	4·9262699	4·9100586	4·8917361
35	4·9176545	4·9167275	4·9172834	4·9131269	4·9009276	4·8846773
40	4·8983099	4·8957153	4·8957153	4·8957153	4·8878297	4·8741489
45	4·8778790	4·8727715	4·8708221	4·8726488	4·8690415	4·8584453
50	4·8563008	4·8477185	4·8420910	4·8420910	4·8420910	4·8350227
55	4·8335108	4·8203623	4·8089306	4·8016086	4·8034322	4·8000869
60	4·8094410	4·7904912	4·7706578	4·7479786	4·7479786	4·7479786
65	4·7840195	4·7578740	4·7264845	4·6769307	4·6684337	4·6702568
70	4·7571703	4·7222582	4·6755010	4·5828081	4·5543316	4·5543316
75	4·7288134	4·6833682	4·6166573	4·4581167	4·3906588	4·3814239
80	4·6988640	4·6409030	4·5487417	4·2929285	4·1558802	4·1235250
85	4·6672328	4·5945338	4·4703556	4·0740910	3·8191047	3·7388571
90	4·6338252	4·5439018	4·3798847	3·7841802	3·3360209	3·1651084
95	4·5985415	4·4886151	4·2754657	3·4001132	2·6430667	2·3093376
100	4·5612764	4·4282458	4·1549486	2·8913103	1·6490665	1·0329186

Number Living.

Age.	Experience Table.	Gompertz's Curve passing through Three Ages.					
	L.	10, 20, 30.	20, 30, 40.	30, 40, 50.	40, 50, 60.	50, 60, 70.	60, 70, 80.
10	100,000	100,000	99,552	97,471	89,867	84,355	80,019
15	96,636	96,667	96,497	95,136	88,988	83,937	79,757
20	93,268	93,268	93,268	92,510	87,837	83,342	79,367
25	89,835	89,808	89,866	89,570	86,336	82,494	78,790
30	86,292	86,292	86,292	86,292	84,386	81,294	77,936
35	82,581	82,728	82,552	82,658	81,870	79,603	76,679
40	78,653	79,124	78,653	78,653	78,653	77,238	74,843
45	74,435	75,488	74,606	74,271	74,585	73,968	72,185
50	69,517	71,829	70,424	69,517	69,517	69,517	68,395
55	63,469	68,157	66,125	64,407	63,330	63,596	63,108
60	55,973	64,482	61,729	58,974	55,973	55,973	55,973
65	46,754	60,816	57,263	53,270	47,526	46,605	46,801
70	35,837	57,170	52,754	47,370	38,266	35,837	35,837
75	24,100	53,557	48,236	41,367	28,716	24,584	24,067
80	13,290	49,988	43,742	35,379	19,630	14,318	13,290
85	5,417	46,476	39,313	29,536	11,860	6,593	5,481
90	1,319	43,035	34,987	23,982	6,084	2,168	1,463
95	89	39,677	30,805	18,857	2,513	440	204
100	0	36,415	26,807	14,287	779	45	11

The curves corresponding to these last are exhibited in the diagram appended to this paper.

At the earlier ages, the three points, as shown in the diagram, are nearly in a straight line and indicate but little curvature, and in these cases the curve which is made to pass through them resembles that of an ordinary logarithmic curve, asymtotic with the line of abscissas, is comparatively rectilinear, and, for the latter ages, exhibits too great a longevity. When the calculated curve coincides at later ages, and comes into nearer juxtaposition towards the end of life, it will be seen that it then falls considerably below the principal curve at the earlier ages. It may be further noticed that, during any portion of life, one or other of the curves will present a tolerable approximation to the original one for a period of about thirty years. In all the curves a contrary flexure occurs near the age of 70, which is a necessary consequence of the decrement of life attaining its greatest value and afterwards diminishing.

The few remaining observations I have to make have reference to a short and comprehensive paper by Professor De Morgan, inserted in the October Number of the *Journal*, p. 27, in which he deduces Simpson's rule from Gompertz's law of mortality. This paper is very interesting, and the perusal of it has led me to a more extended examination of the subject, which I do not hesitate to lay before the members of the Institute, as the result presents

itself in a manner so marked and decided. Professor De Morgan has, with his usual ability and elegance, proved the accuracy of Simpson's rule on the assumption of Gompertz's law. He has furthermore stated that the same conclusion might be arrived at from a yet more generally expressed law, of which that of Gompertz would appear to be only a particular case. In the following brief discussion of this point, I propose to show that the seeming generality of this last assumption, although apparently a very feasible superstructure, has in reality no foundation whatever, and that the truth of Simpson's rule, which is a necessary consequence of Gompertz's law, cannot exist on any other hypothesis. I allude specifically to the last paragraph of Professor De Morgan's paper, which is as follows:—

"It is not, of course, necessary that the progression of powers should be precisely that of Mr. Gompertz, and the following theorem may easily be demonstrated. If a_n be the chance of a life living n years, and

$$a_n=(a_1)^{p_n},$$

where p_n is not a function of a, then the annuity on any number of lives is not altered in value if, instead of any part of those lives, a single life of equivalent value be substituted."

On first reading the announcement of this theorem, it appeared to me that the function of n, denoted by p_n, instead of being arbitrary, or possessing any considerable amount of generality, was, from the nature of the subject, very much restricted in its application. For, since $a_0=1$, it is obvious that, when n is taken equal to 0 and 1 respectively, the equation must then determine $p_0=0$, $p_1=1$, thus necessarily fixing two values; and if the inquiry be pushed a little further, it will be found that the conditions imposed upon the function are such that it can indeed represent neither more nor less than Gompertz's law, as I shall now proceed to show.

For convenience of notation, let the function p_n be designated by fn; then, since $a_1=\frac{L_{a+1}}{L_a}$, and $a_n=\frac{L_{a+n}}{L_a}$, the proposed equation may be otherwise written thus:—

$$\frac{L_{a+n}}{L_a}=\left(\frac{L_{a+1}}{L_a}\right)^{fn} \quad . \quad . \quad . \quad . \quad (1)$$

Or, dividing by $\frac{L_{a+1}}{L_a}$,

$$\frac{L_{a+n}}{L_{a+1}}=\left(\frac{L_{a+1}}{L_a}\right)^{fn-1} \quad . \quad . \quad . \quad . \quad (2)$$

And as these equations are general, and the value of fn is independent of the age a and varies only with n, by putting $a+1$ for a in (1) we get

$$\frac{L_{a+1+n}}{L_{a+1}}=\left(\frac{L_{a+2}}{L_{a+1}}\right)^{fn}.$$

Also, by making $n=2$ in (2),

$$\frac{L_{a+2}}{L_{a+1}}=\left(\frac{L_{a+1}}{L_a}\right)^{f(2)-1}.$$

Therefore, by substitution,

$$\frac{L_{a+1+n}}{L_{a+1}}=\left(\frac{L_{a+1}}{L_a}\right)^{fn[f(2)-1]} \quad . \quad . \quad . \quad (3)$$

But, by putting $n+1$ for n in (2), we have

$$\frac{L_{a+n+1}}{L_{a+1}}=\left(\frac{L_{a+1}}{L_a}\right)^{f(n+1)-1} \quad . \quad . \quad . \quad (4)$$

Whence, equating the values in (3) and (4), we deduce the following relation for determining the nature of the function characterised by the symbol f, viz. :—

$$f(n+1)-1=fn\{f(2)-1\},$$
$$\text{or } f(n+1)=1+q.fn \quad . \quad . \quad . \quad . \quad (5)$$

in which $q=f(2)-1$ may be treated as a constant factor, being independent of n. The only solution this functional equation admits of is—

$$fn=\frac{q^n-1}{q-1} \quad . \quad . \quad . \quad . \quad (6)$$

This, indeed, is readily ascertained by rigid induction; for, by making, in equation (5), n successively equal to the consecutive numbers 1, 2, 3, &c., we at once find

$$\begin{aligned} f(2)&=1+q \\ f(3)&=1+q.f(2) \\ &=1+q+q^2 \\ f(4)&=1+q.f(3) \\ &=1+q+q^2+q^3 \\ &\text{\&c.} \qquad \text{\&c.} \\ f(n)&=1+q.f(n-1) \\ &=1+q+q^2 \ldots\ldots +q^{n-1} \\ &=\frac{q^n-1}{q-1}. \end{aligned}$$

The expression originally assumed for a_n therefore becomes

$$a_n=(a_1)^{\frac{q^n-1}{q-1}}, \text{ where } q=a_2-1;$$

and may be written

$$\frac{L_{a+n}}{L_a}=\left(\frac{L_{a+1}}{L_a}\right)^{\frac{q^n-1}{q-1}} \quad . \quad . \quad . \quad . \quad (7)*$$

* By making $\frac{L_1}{L_0}=10^{-\beta}$, the formula (7) gives

$$\frac{L_x}{L_0}=10^{-\beta\frac{q^x-1}{q-1}},$$

which is by far the most convenient form for calculations in general.

If the age a be taken at birth, or $a=0$, then

$$L_n = L_0\left(\frac{L_1}{L_0}\right)^{\frac{q^n-1}{q-1}},$$

which is equivalent to

$$L_n = k(g)^{q^n} \quad . \quad . \quad . \quad . \quad (8)$$

$$\text{where } g=\left(\frac{L_1}{L_0}\right)^{\frac{1}{q-1}}, \quad \text{and } k=\frac{L_0}{g}, \quad \text{also } q=\frac{\Delta \log L_1}{\Delta \log L_0};$$

and thus we find the law of mortality to be of precisely the form of Gompertz.

It also follows obviously from what has been established, that Gompertz's law of mortality may be directly obtained by assuming, as a hypothesis, the accuracy of the rule by which a status of equivalent value is substituted in the calculation of a joint annuity on several lives.

I have only further to remark, that as the proposed equivalence subsists with respect to each of the several terms of the series representing the annuities, it will obviously hold good whether the annuity be for a term of years or otherwise. This consideration is by no means unimportant, for it has been shown that Gompertz's formula is capable of exhibiting the true law of mortality with sufficient accuracy for long periods, and the exact determination of the value of an annuity on three or more lives for a term of years involves a considerable amount of calculation. It may, therefore, be sometimes desirable to know under what circumstances the convenient rule of substitution can be judiciously applied, and the degree of accuracy that may be expected from it.

On the Tendency of some Systems of Distribution of Surplus to defeat the Object of Life Assurance. By James Terry, *Esq., of the Hand in Hand Insurance Society.*

[Read before the Institute, 27th January, 1862, and printed by order of the Council.]

The system pursued by each Office in the distribution of surplus may be divided into three parts, namely—

1. The class of members amongst which surplus is distributed;
2. The actual distribution of it amongst that class; and
3. The interval which elapses between successive distributions.

It is impossible to obtain a comprehensive view of the subject without considering separately these three distinct parts; and, since it forms no part of the present purpose to estimate the relative advantages of different Offices, no attempt will be made to follow the numerous combinations or systems of distribution produced by differences in practice under each of the above heads.

There is no doubt that the methods employed in the distribution of surplus are such as to render the scale of premiums paid in many instances inequitable. Whether we look to the formation of the class amongst which surplus is distributed, or to its actual distribution, or to the interval between successive distributions, there are inequalities in each which nothing but a fortuitous balance of errors can prevent from producing such a result. That such balance seldom obtains will be evident when it is borne in mind that the general tendency of those inequalities is in the same direction. The introduction of an equal period of non-participation for all ages at starting is manifestly an advantage to those who enter at the younger ages; and it has been shown, and will be incidentally exhibited presently, that the actual distribution of surplus tends to benefit the same class, whilst the introduction of a period of several years between successive distributions must be regarded as a series of advantages to the younger entrants, similar to that conferred upon them by the equal period of non-participation at starting. So that, upon the whole, it may fairly be said, that under present systems there is, generally speaking, a pecuniary advantage to be derived from a resort to life assurance when young. This part of the subject has been already treated by several Members of the Institute and others, and it is not my intention to pursue the inquiry as to the equality or inequality of the scale of premiums paid. Any discrepancy under this head may, for the future, be avoided by an alteration in that scale. But there is another inequality in some of the present systems which is essential to them, and which no scale of fixed premiums can remove. However accurately the interest of different members may be adjusted at starting, there is an inequality which develops itself afterwards, and which is the more objectionable, since its tendency is to defeat life assurance by the introduction of the principle of a tontine to an extent that cannot fail to impair the usefulness of many important institutions.

1. *The class of members amongst which surplus is distributed.*

Notwithstanding that membership in a Mutual Office itself creates the right of participation in surplus, the recognition of this right is invariably withheld, either until a certain number of premiums have been paid, or else until those premiums at interest amount to the sum originally assured. The effect of this will be easily seen. If death occur during the period of non-participation, the whole of the surplus belonging to the assured is forfeited; the Office is thus, so far as that amount is concerned, converted into a lottery, in which the survivors of a certain period after admission take all, and those who happen to die in the interval lose all. That portion of the premium which is required to provide the sum assured is applied to carry out the principle of life assurance, and the remainder to destroy it. Of course, the greater the contribution to surplus, or, in other words, the higher the scale of premiums paid and the longer the period of non-participation, the more injurious will be the effect; and in an Office in which participation is withheld until the premiums at interest amount to the sum originally assured, it will be seen how great must be the displacement of life assurance by the introduction of a tontine in the formation of the class amongst which surplus is distributed.

2. *The actual distribution of surplus.*

The original allotment resulting from the particular method employed for the purpose, and the ultimate application of the amounts so allotted, are, of course, distinct operations. Whether the bonus upon a policy be applied to the assured in the form in which it was allotted, or whether that form be altered, is, in point of principle, immaterial. At the same time, as remarked by Mr. Pinckard, the option of receiving the allotted bonus, either in cash or in a reversion, can only be given at the time of issuing the policy. It is for the assured to determine beforehand whether he wishes to secure payment of a fixed sum at death or to lay out a certain sum in premium, otherwise the adjustment in the former case of the premium to the sum assured, or, in the latter, of the sum assured to the premium, which takes place subsequently, and which proceeds upon a total disregard of the state of health of 'the life,' could not properly be carried out without a medical examination.

The present object is to exhibit the original allotment of surplus according to the following six methods, namely:—

1. A reversion in ratio of amount originally assured and number of years since commencement.
2. A reversion in ratio of amount actually* assured and number of years since the previous valuation.
3. A reversion in ratio of total amount of premium paid.
4. Cash in ratio of total amount of premium paid, accumulated at interest.
5. Cash in ratio of value of policy.
6. Cash in ratio of excess of (4) over (5).

With the exception of the first two, all the above more or less involve the scale of premiums charged. In calculating the distribution of surplus by means of them, regard must be had to this circumstance. But there are are so many rates of premium in use, that, out of more than 100, there are not half a dozen alike. However, for the most part, the difference between them is so slight, that the following scale, which is the average of them all, will be found to be exceedingly near the greater number:—

20 . . £1 18 7	40 . . £3 4 8	60 . . £6 18 6
30 . . 2 9 2	50 . . 4 10 7	

Assuming this scale to have been employed throughout, Table I. shows the ratio of distribution between policies on lives of different ages at entry after the lapse of five years by each of those methods; and since it is necessary, for the purpose of comparison, that all the results should be exhibited in the same form, and since surplus is always ascertained in cash, the reversionary bonuses produced by the first three are rendered in their equivalent present values.

That these ratios, exhibited after the first five years, are not materially affected by lapse of time, but will nearly obtain at each succeeding distribution, may be inferred from Table II. showing as before, except that 25 years from admission, instead of five only, are supposed to have elapsed.†

Having exhibited the ratios of distribution so far as they are affected by the ages of the lives only, it becomes necessary to determine them so far as they are affected by the ages of the policies only, because, in point of fact, as will presently appear, the division of surplus is influenced by both these circumstances, and by the latter much more than by the former. If the five policies

* Supposing no bonuses to have been surrendered.

† Thus it will be seen that, as between policies on lives of different ages at entry, but of the same length of standing, the remark of Mr. Jellicoe (*Assurance Magazine*, vol. iii., p. 191), with reference to No. 6 method, that "at each septennial division the same degree, or very nearly the same degree, of injustice is perpetrated as at the first," confirmed, and, indeed, may be extended to all the other methods here spoken of.

in Table I. be supposed to constitute a class, and a distribution to take place among eight such classes, differing from each other only in length of standing—thus, one having been assured 5, another 10, another 15, and so on up to 40 years—Table III. will show the ratio of such distribution by each of the foregoing methods.

The original allotments of surplus resulting from the employment of the foregoing methods having been exhibited, it remains to compare them with those which would have attended a distribution upon recognized principles. It has, indeed, been contended that all appropriation during the currency of a policy is wrong, but practically the correctness of this argument has certainly not been admitted. It is true that the surplus upon a policy depends upon the rates of mortality and interest, and other circumstances, which may prevail throughout its entire continuance, and cannot, therefore, be exactly determined until the policy itself has ceased; but does it follow, therefore, that until then all distribution is wrong? The cost of assuring a sum to be paid at death cannot be ascertained beforehand, but, since the uncertainty as to what that cost will amount to may, for all practical purposes, be regarded as existing only between comparatively narrow limits, Offices have always undertaken the assurance of a particular sum in consideration of the receipt of a fixed premium; and this has been accomplished by stipulating for such an amount of premium as may be regarded as safe to cover the risk incurred, notwithstanding its uncertainty. Without thus relying upon probability, and acting upon a certain degree of security, it is difficult to see how mutual assurance could be carried out; and it is demonstrable that an annual division of surplus upon a policy may be made without lessening that degree of security. For instance, suppose an Office to undertake assurances upon a deposit of 1 per cent. per annum upon the sum assured, in addition to the probable premium, and, at the end of the first year, it be found that the deposit for that year has not been required, where is the error in refunding it, or in remitting the next and every future year's stipulated deposit so long as such an amount remains? If the risk in respect of which a deposit was required for the first year does not materially alter in subsequent years, why should the reserve to meet it be doubled for the second year, trebled for the third, and so on?

Although the *amount* of surplus existing in an Office at the time of making a valuation depends upon the rates of mortality and interest, and other circumstances, which have prevailed in it, yet the *ratio* in which that surplus has been contributed by the

different members, and therefore the ratio in which it should be distributed among them, is clearly independent of those rates, and is determined by the difference between the probable premium and that actually paid at each age.* Employing the Combined Experience class mortality, $3\frac{1}{2}$ per cent., as calculated by Mr. Higham, with a constant addition to the net premium of 5*s*. for expenses, the following exhibits the probable premium for assuring £100:—

20	1·695	40	2·751	60	5·766
30	2·132	50	3·881		

and if the difference between these amounts and the premium actually paid respectively be taken and adjusted to the scale employed in Table I., it will exhibit the ratio in which surplus should be distributed, as shown in column 7 in that table; and this ratio being evidently constant, will apply equally after the lapse of 25 years, as in Table II., column 7, or after any other number of years.

Applying the same test to Table III., it is evident that, as the amount of its contribution to the surplus, accrued since the previous valuation, is not at all affected by the number of years during

* Of course this is true only upon the supposition here made, that the policies are for an equal amount and have been an equal number of years in force since the previous valuation. The share of each contributor being represented by the well-known formula $\phi a^n . \frac{\mathrm{S}}{f.\phi a^n}$, it is evident that the ratio between the amounts so represented will not be affected by dividing the above expression by what, under such circumstances, will be the constant quantity $a^n . \frac{\mathrm{S}}{f.\phi a^n}$.

Referring to this method, an able writer has remarked (see paper by Mr. Sprague, in *Assurance Magazine*, vol. vii., "On certain methods of dividing the surplus among the assured in a Life Assurance Company, and on the rates of premium that should be charged to render them equitable"), that, "although it is as equitable as any that could be devised, there seems to be one point in its working which is open to some objection—I mean, that, since the successive cash bonuses on each policy will be of about the same magnitude, supposing the state of the Company to remain the same, it follows that the successive reversionary bonuses will form a decreasing series, on account of the increasing age of the life assured. This, of course, is no objection to the theory of the method; but I think it will be admitted, that the public at large, who understand nothing of the theory of life assurance, expect that each reversionary bonus on their policies should be at least as large as the preceding one; and it will probably be thought an advantage if a method of division could be adopted which, while equitable to all parties and simple in its details, should at the same time satisfy the condition mentioned above." The writer then proceeds to indicate two means by which these objects may be, to some extent, attained. The first is, "to distribute only a portion of the surplus that appears as the result of the valuation, and reserve the remainder to accumulate and to be divided at a future time among the survivors of those who have contributed towards it." The second, and, as added, the more practicable one, is to regard the loading as the annual premium for an increasing assurance. In both these cases, however, the effect would be to give an advantage to those who live long, at the expense of those who die soon; and, however desirable it may be to satisfy public expectation, it does not seem to me to be possible for an Office to adopt a system calculated to yield reversionary bonuses of increasing amounts upon a policy, without some departure from the fundamental object of the Association.

which a policy may have been in existence before that period, so the distribution of it should be made regardless of this circumstance, and the amount allotted to each of the classes in Table III. should be the same, whether such class has been assured 10, 20, 30, 40, or any greater number of years, as shown in column 7 in that table.

It is true that, if the surplus in an Office be applied to increase the amount assured upon the same data as those employed in determining the premium, the assured will be entitled to participate in respect of such increase at subsequent divisions. On the other hand, if the same application be made upon "true" or probable data, this will not be the case; but here, the sum assured being increased without any corresponding addition being made to the "loading" or "marginal guarantee," the effect will be to gradually lessen that degree of security under which the Office is acting.

The tabular results given, so far as they involve the scale of premiums paid, would, of course, be affected by an alteration in that scale; but this circumstance would operate less strongly with reference to Table III. Altogether, perhaps, the following conclusions upon the foregoing six methods are warrantable:—

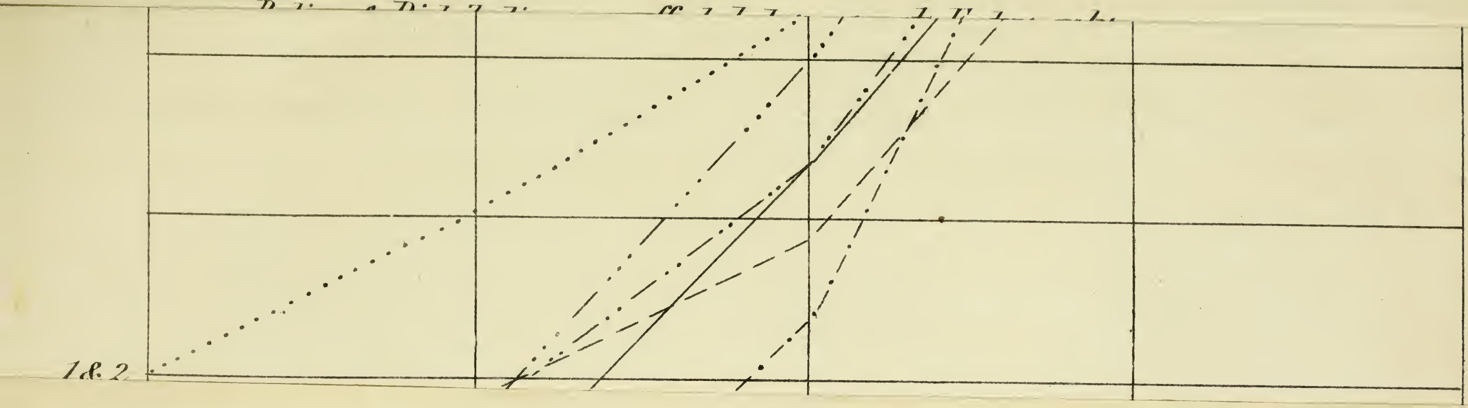
1. As regards the ages at entry of the lives—That there is a general tendency in favour of young ones.
2. As regards the ages of the policies—That there is a general and very powerful tendency in favour of old ones.

When these favourable tendencies are combined, as in the case of an old policy originally effected upon a young life, it will be seen to what an extent the displacement of life assurance must be carried out in the actual distribution of surplus. For instance, Table III. shows that an Office employing the first method would allot to a class that had been assured 40 years twice as much as it would be entitled to (£254, instead of £125 only, out of each £1,000 distributed); and it may be gathered from Tables I. and II., that the amount so allotted would be divided between the policies composing that class in such a manner that a policy originally effected upon a life of 20 would receive five times its proper amount. Had Table III. been calculated for more than 40 years—and by the mortality assumed it might have been continued for the youngest lives for 40 years longer—the results would have been much more unfavourable.

3. *The interval which elapses between successive distributions.*

All deferment in the distribution of surplus—involving, as it does, the absorption of a larger amount of capital in carrying on

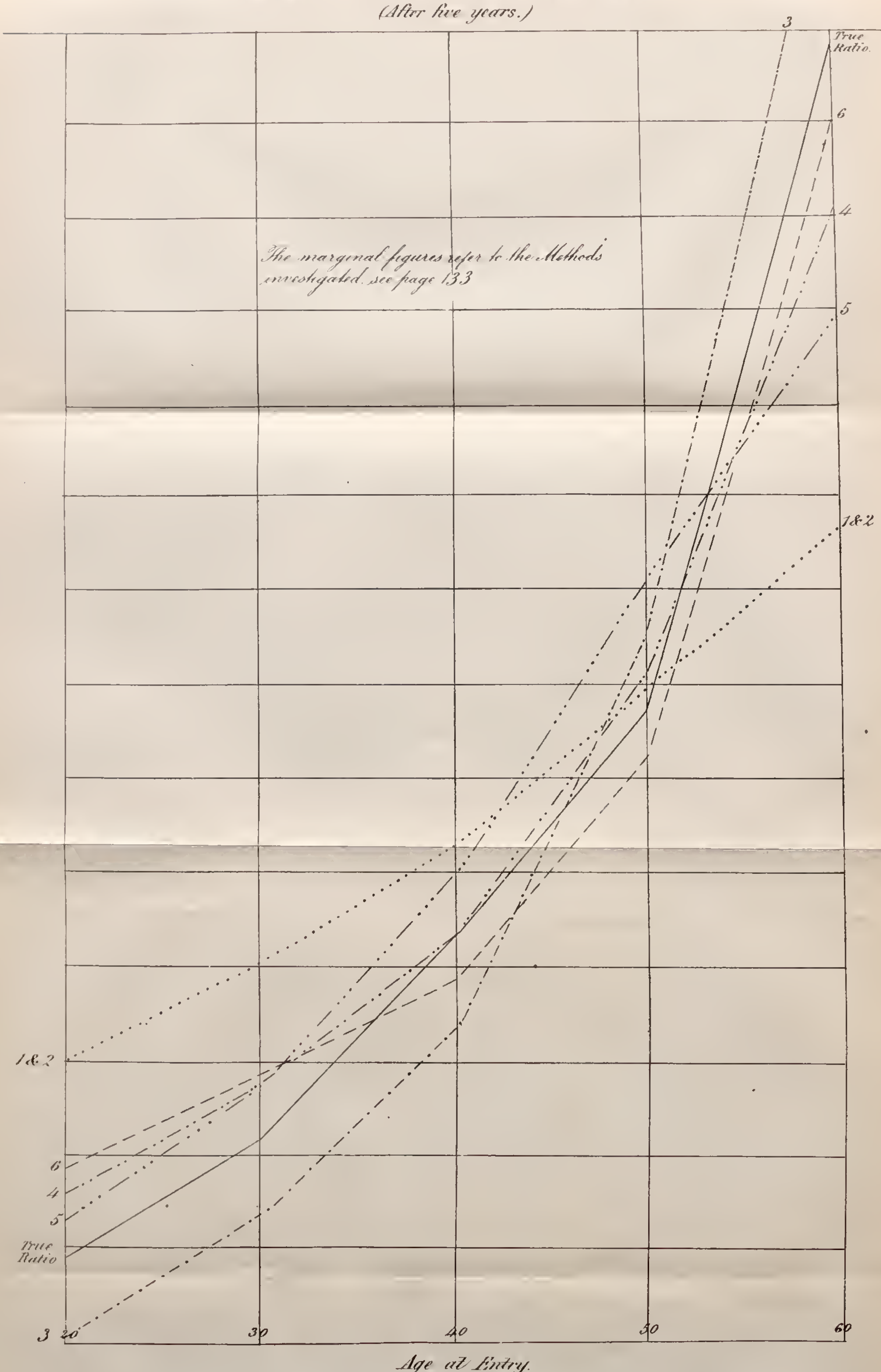
No. 1.

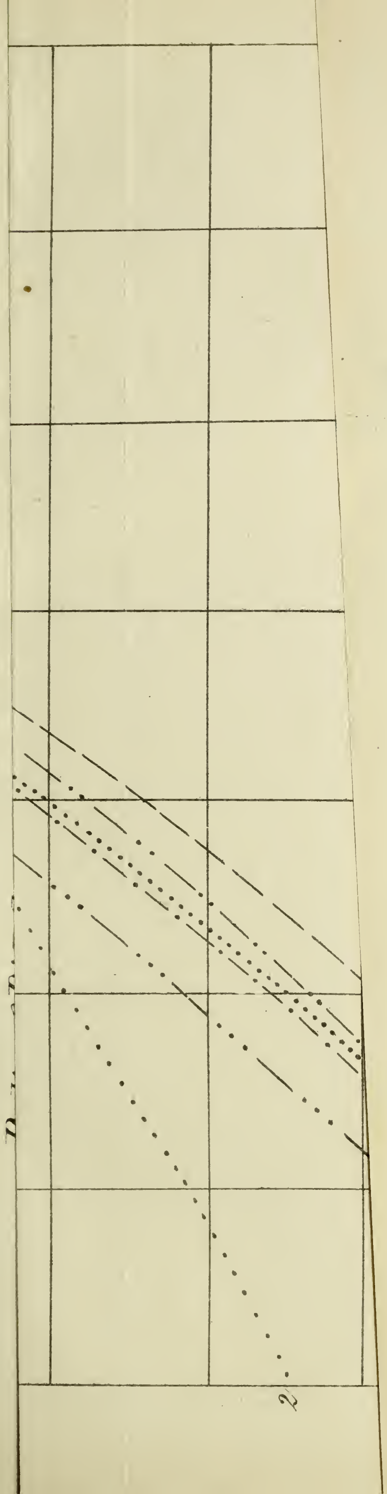


No. 1.

Ratio of Distribution as affected by age at Entry only.—

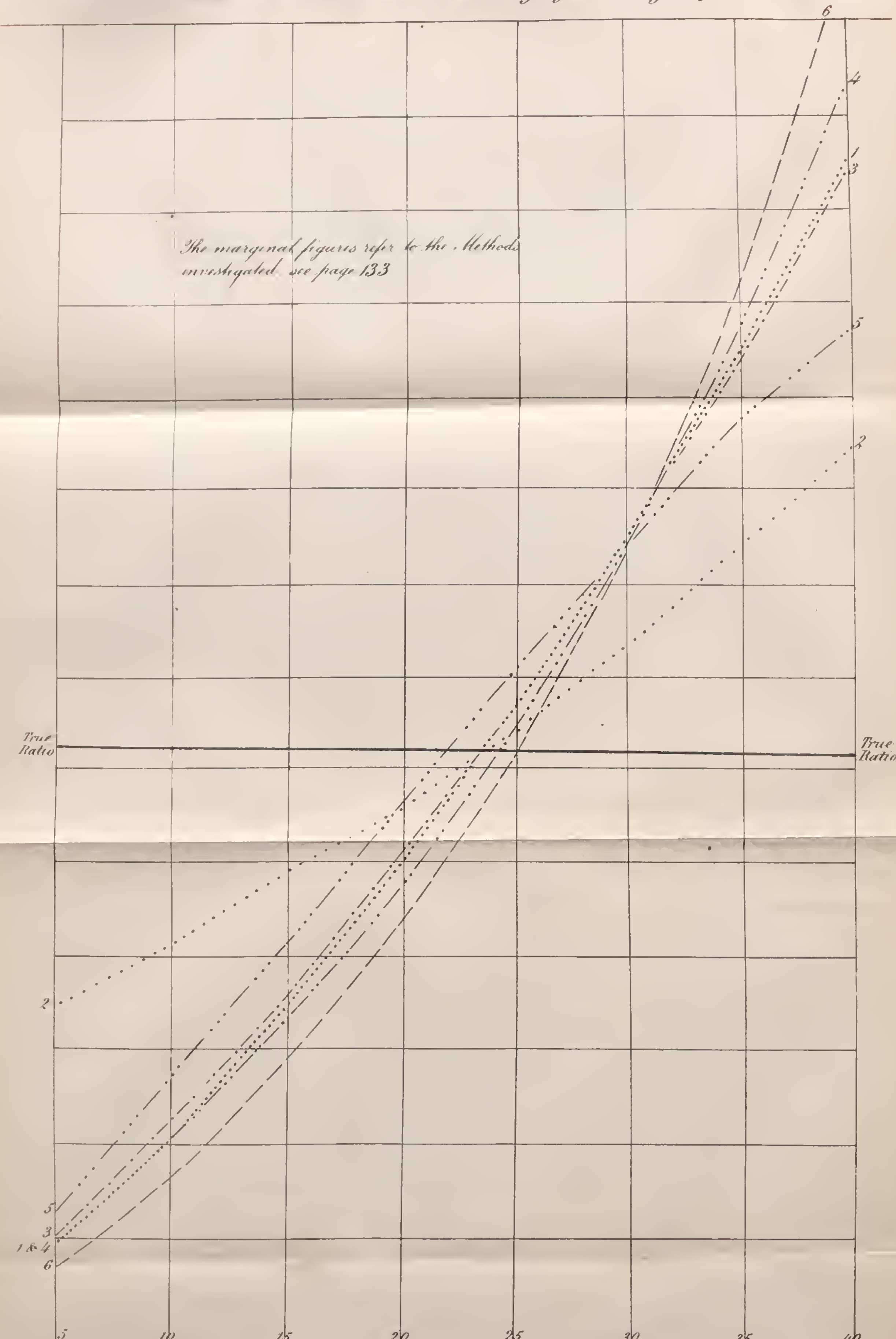
(After five years.)





No. 2.

Ratio of Distribution as affected by age of Policy only.



the business of life assurance than is necessary—is inexpedient. This objection may be urged most strongly against deferring all distribution until a policy has ceased, and attaches, though with less force, to the practice of introducing intervals of 3, 5, 7, and 10 years between successive distributions. Even in its present limited operation, the practice alluded to must alone have encumbered life assurance with a very large amount of useless capital. But its effect is not simply to cause the allotment of surplus to be deferred. The amount accrued during the interval after the first distribution is divided among the survivors of that interval only, and is lost to the assured if the life drops. Those who survive take all, and those who happen to die lose all. This is repeated at every subsequent interval, and thus the same opposition to the principle of life assurance which was created by the introduction of a period of non-participation at starting, is effectually maintained throughout the existence of the policy.

An annual distribution is free from both these objections. It avoids the imposition of a burden which has not only impeded the progress of life assurance, but occasioned considerable abuse of it. It is the obvious and legitimate period which should separate successive divisions.

Although not necessarily involving an annual valuation, some practical objection to an annual distribution may, perhaps, be urged, from a consideration of the labour it would entail, but there is no doubt that the subject admits of considerable simplification. So long as the ratio of distribution coincides with the ratio of contribution, it is not material which determines the other, and there seems to be no reason why the latter should not be made conformable to a convenient mode of distribution. For instance, if the amount of contribution to surplus were the same at each age, one single operation would determine the percentage of allotment, after which the present value of the bonus to be assigned to each policy would be little more than a matter of mere inspection.

However attractive may be its advantages when a resort to life assurance is first made, there is no doubt that this attraction loses much of its force with the assured when by and bye he ceases to be beneficially within its operation. So long as there is a chance of the premature failure of the life, its claims are not likely to be forgotten, but when the continued existence of that life has denied to the assured all pecuniary advantage, and committed him to an indefinite pecuniary sacrifice, he is apt to regard the matter only

in the light of a bad bargain, to be made the best of. In a Proprietary Company the existence of such a feeling, however unprincipled it may be, is not attended with any dangerous consequences, for the assured have no opportunity for its gratification; and where the principle demands that a loss should be sustained, escape from it is rendered impossible. But in a Mutual Society the case is different. The existence of a fund of which the assured are themselves the distributors, the absence of a knowledge on their part of the principles by which they should be governed as such, to which may be added the plausibility of the arguments by which its distribution in favour of the older members is maintained, offer a temptation to the latter to lessen their liability, which has already, on more than one occasion, to some extent been found irresistible. Hence the importance of offering the strongest opposition to the growth of such a feeling in a Mutual Office. How far it prevails in them at the present time it may be difficult to say, but its existence is clearly to be traced in, and it is to be feared that it derives some encouragement from the foregoing methods employed in, the distribution of surplus.

Having considered the three parts which together may be regarded as constituting an entire system of distribution as at present carried out, and having shown that there is a tendency in each to defeat the purpose it is employed to subserve, it may be sufficient to refer to actual results, in order to gather some idea of their *combined* effects.

The following is based upon published statements:—

(1) Sum payable at Death for every £100 originally assured.	(2) Age of the Policy.	(3) Youngest Age at Entry at which Total Premium paid, accumulated at 4 per Cent., would have amounted to the Sum in Column 1.
	Years.	Years.
654·2	65	19
588·2	61	23
555·7	59	24
527·2	57	26
498·7	55	28
470·2	53	29
441·7	51	30
415·7	49	32
392·2	47	33
368·7	45	35

Thus it appears that cases must have occurred in which the sum payable at death has exceeded the accumulated premiums and

4 per cent. compound interest, notwithstanding that the life had far outlived its average after lifetime when assured. It must be admitted that these results have been produced under somewhat exceptional circumstances; but, after making due allowance for such, they yet afford a strong illustration of the extent to which its original intention may and, indeed, must be defeated in an Office which allots to its policies periodical bonuses of continually and rapidly increasing value.

It is, perhaps, premature to expect to find evidence equally strong of the injurious systems pursued elsewhere, because sufficient time can hardly have elapsed to allow of the existence of many very old policies under them—an impediment which is being daily removed; but already, in several instances, the sum payable at death has attained a magnitude equal to nearly 300 per cent. upon the amount originally assured—and this, too, even in cases in which a portion of the profits is handed over to proprietors.

The following may be taken as specimens of the values of *periodical* bonuses allotted at recent distributions for every £1,000 originally assured:—

(1) Value of Bonus allotted.*	(2) Premiums in respect of which Bonus was allotted, accumulated at $3\frac{1}{2}$ per Cent.	(3) Excess of (1) over (2).
948·8	264·5	684·3
883·1	264·5	618·6
328·3	264·5	63·8
293·0	273·4	19·6
277·5	252·9	24·6
266·2	229·7	36·5
264·7	210·3	54·4
246·7	201·6	45·1
225·3	180·1	45·2
169·4	114·0	55·4
160·2	169·4	− 9·2

It is impossible that the bonuses assigned to the above policies can fairly have accrued to them during the interval since the previous valuation. No; the explanation is—that the systems pursued are such that the bonuses apportioned to a policy during its earlier years are inadequate—that this inadequacy is compensated for by the excess of the subsequent ones—and that the latter are rendered still more excessive from the fact that, in many instances, a policy

* Mr. Higham's "Mixed Mortality Lives from among whom the effect of selection has been exhausted." Interest $3\frac{1}{2}$ per cent.

drops before such compensation is made. So that whether or not the assured receives the amount of bonus to which he is entitled, in a great measure depends upon the number of years his policy remains in force; and it is this inequality which is essential to many of the systems generally pursued, and will remain, although the scale of premiums paid, when taken in connection with the benefits to which they entitle the assured, is altogether free from objection.

Enough has been said to show that the whole system of distribution of surplus is strongly pervaded with the pernicious properties of a tontine. The fact is not attempted to be disguised; on the contrary, in the prospectuses issued by some Offices it is brought prominently forward as an additional inducement to the public to enter them; and extreme cases, in which the very object for which they were established has been almost if not altogether defeated, are paraded as evidence of the immense success that has attended the working of those institutions. At the present time especially, when an attempt is being made to revive this injurious form of lottery, the importance of the subject, which has been but imperfectly handled in the present paper, is such as to claim the attention of all who wish to see the principle of life assurance carried out in its integrity.

The total amount assured in this country may, perhaps, be accepted as evidence of the extent to which the public have sought to avail themselves of the advantages of that principle, but it fails as an index of the extent to which those advantages have been secured to them. And this must continue to be the case so long as the transfer from one class to another, which the payment of one portion of the premium effects, is followed by a corresponding re-transfer resulting from the application, or rather the misapplication, of the remainder.

It should always be borne in mind, that in all cases in which the duration of "the life" from the period of its selection coincides with the average duration of life, the principle of life assurance is inoperative; that its usefulness should be in proportion as that average is departed from; and that in extreme cases, either of long or short life, its importance should be most strongly illustrated. It is in these latter that the foregoing methods employed in the distribution of surplus are most defective.

In conclusion, I may refer to the two diagrams appended to this paper for further illustration of the effects of the methods exhibited in Tables I. and III.

TABLE I.—*Showing the Ratio of Distribution of Surplus between Policies on Lives at different Ages at Entry, after the Lapse of Five Years, by each of the following Methods:—Upon the Combined Experience Class Mortality,* $3\frac{1}{2}$ *per Cent., as calculated by Mr. Higham (except Columns* 4, 5, *and* 6, *in which* 3 *per Cent. and a Rate of Mortality approximately yielding the Scale of Average Premiums payable are employed).*

Age when Assured.	Reversion in Ratio of			Cash in Ratio of			Ratio of Contribution to Surplus.
	Amount originally Assured, and Number of Years since commencement.	Amount actually Assured, and Number of Years since the previous Valuation.	Total Amount of Premium paid.	Premiums Accumulated at Interest.	Value of Policy.	Excess of (4) over (5).	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
20	136	136	62	101	94	107	82
30	161	161	93	129	128	130	114
40	193	193	146	170	185	157	170
50	233	233	248	237	262	216	228
60	277	277	451	363	331	390	406
	1,000	1,000	1,000	1,000	1,000	1,000	1,000

TABLE II.—*Showing the Ratio of Distribution of Surplus between Policies on Lives at different Ages at Entry, after the Lapse of* 25 *Years, by each of the following Methods:—Upon the Combined Experience Class Mortality,* $3\frac{1}{2}$ *per Cent., as calculated by Mr. Higham (except Columns* 4, 5, *and* 6, *in which* 3 *per Cent. and a Rate of Mortality approximately yielding the Scale of Average Premiums payable are employed).*

Age when Assured.	Reversion in Ratio of			Cash in Ratio of			Ratio of Contribution to Surplus.
	Amount originally Assured, and Number of Years since commencement.	Amount actually Assured, and Number of Years since the previous Valuation.	Total Amount of Premium paid.	Premiums Accumulated at Interest.	Value of Policy.	Excess of (4) over (5).	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
20	147	147	68	101	111	95	82
30	172	172	103	129	146	120	114
40	202	202	158	170	190	159	170
50	229	229	251	237	242	235	228
60	250	250	420	363	311	391	406
	1,000	1,000	1,000	1,000	1,000	1,000	1,000

Table III.—*Showing the Ratio of Distribution of Surplus between Classes of Policies of different Lengths of Standing, from 5 to 40 Years, by each of the following Methods:—Upon the Combined Experience Class Mortality, 3½ per Cent., as calculated by Mr. Higham (except Columns 4, 5, and 6, in which 3 per Cent. and a Rate of Mortality approximately yielding the Scale of Average Premiums payable are employed).*

No. of Years Assured.	Reversion in Ratio of			Cash in Ratio of			Ratio of Contribution to Surplus.
	Amount originally Assured, and Number of Years since commencement.	Amount actually Assured, and Number of Years since the previous Valuation.	Total Amount of Premium payable.	Premiums Accumulated at Interest.	Value of Policy.	Excess of (4) over (5).	
	(1)	(2) *	(3)	(4)	(5)	(6)	(7)
5	19	70	20	19	27	15	125
10	42	83	44	41	55	34	125
15	69	97	71	66	83	58	125
20	100	112	101	95	113	87	125
25	134	129	134	129	142	123	125
30	171	148	171	169	169	168	125
35	211	170	209	214	196	223	125
40	254	191	250	267	215	292	125
	1,000	1,000	1,000	1,000	1,000	1,000	1,000

On the Principles on which the Funds of Life Assurance Societies should be Invested. By Arthur Hutcheson Bailey, *Esq., Actuary of the London Assurance.*

[Read before the Institute, 24th February, 1862, and printed by order of the Council.]

Of the many valuable contributions by which the transactions of this Institute have been enriched by the labours of Mr. Samuel Brown, not the least interesting is the paper on the investments of Assurance Companies, read before the Institute on the 25th January, 1858. I feel much diffidence in again reverting to this question; but as Mr. Brown, at the outset of his paper, admits that the subject is too extensive to be exhausted by a single discussion, it has occurred to me that an attempt to arrive at some general principles on which the funds of Life Assurance Societies should be invested, and to consider how such principles can best be reduced

* Supposing a permanent rate of addition of £1. 10s. per cent. per annum.

to practice, might, perhaps, be the means of exciting some useful discussion.

Of the two main elements on which all life assurance transactions depend—the rate of mortality and the rate of interest—the latter, I think, affords more scope for the exercise of judgment and skill than the former. From my own observation and experience, I incline to the belief, that, while there are probably some not inconsiderable variations in the rates of mortality experienced by different Offices, these variations are due less to what is called (not very accurately) the selection of lives—the effect of which, I think, is generally over-estimated—than to the differences of class from which the selection has to be made. However this may be, as far as the mortality is concerned, there can be no doubt that the amount of interest realised on the assets can be materially influenced by the degree of judgment and knowledge brought to bear upon the subject. Mr. Brown has given some apt illustrations of how considerably the accumulation of capital is affected by very small differences in the rate of interest.

From the nature of the business of Life Assurance Societies, their financial arrangements can be made under some material advantages, which it is important to keep clearly in view. They engage to pay fixed sums of money at periods generally long distant from the time when the contracts are entered into: in mercantile phraseology, they may be said to accept bills drawn at very long dates. Unlike Fire Insurance Companies, where the profits of years are sometimes swept away by the disasters of as many days, the probable amount of demands on their resources can be calculated, from time to time, within not very wide limits. To such an extent is this the case, that, notwithstanding all the disturbing effects that war, famine, and pestilence have produced, I doubt whether the ill success of any life assurance project has ever been truly attributable to excessive mortality. Further: Life Assurance Societies, unlike banks and commercial enterprises generally, are not exposed to sudden or unusual demands on their resources in times of panic and financial difficulty; and the object of their investments is not to obtain income for the purpose of expenditure, but to accumulate capital for the payment of claims.

From a careful consideration of these peculiarities, I think that the main principles to be observed in investing the funds of a Life Assurance Society are the following, viz.:—

1. That the first consideration should invariably be the security of the capital.

2. That the highest practicable rate of interest be obtained, but that this principle should always be subordinate to the previous one, the security of the capital.

3. That a small proportion of the total funds (the amount varying according to the circumstances of each individual case), should be held in readily convertible securities for the payment of current claims, and for such loan transactions as may be considered desirable.

4. That the remaining and much larger proportion may safely be invested in securities that are not readily convertible; and that it is desirable, according to the second principle, that it should be so invested, because such securities, being unsuited for private individuals and trustees, command a higher rate of interest in consequence.

5. That, as far as practicable, the capital should be employed to aid the life assurance business.

The first two principles may, perhaps, be looked upon as mere truisms. But it should not be forgotten that life assurance investments should not always be regulated by the same considerations as trusts for private and family purposes. In the latter, usually the most important matter is to secure a fixed, or, at all events, an undiminishing income—fluctuations in the value of the capital being less inconvenient.

If these principles generally are sound, it follows, I think, that some modes of investment which are commonly resorted to should be avoided. It may seem a bold assertion, but, nevertheless, I believe it to be true, that the English Funds are altogether unsuited for life assurance investments. For income they offer probably the best security the world has yet seen; but with us that is a secondary consideration; the capital, the security of which is our first object, is subject to very inconvenient fluctuations in value. But it will be urged, that, on account of their ready convertibility, some portion of the assets should always be invested in the Funds. This advantage I believe to be more apparent than real. In the practical working of Life Offices it will commonly be found that, in an easy condition of the money market, the funded property has a tendency to increase, although the purchases must, of course, then be made at prices above the average—because, money being plentiful, eligible securities are then scarce. In periods of pressure, on the other hand, when favourable opportunities for investment are abundant, the funded property is practically inconvertible, from the determination not to effect sales at a loss, except in cases

of absolute necessity. I believe that the Assurance Companies are now much less extensive fundholders than formerly. But there is still a prevalent idea that one half the assets should always be invested in the Funds. Upon this rule the wealthiest of the Offices still acts, and the idea has been so fostered by the large profits realised by some of the older Companies from their purchases at the beginning of this century, that it seems necessary to point out that the risk is now in the opposite direction, and that perfect security is very far from being one of the advantages of the Funds.

The same considerations apply to the Preference Stocks of Railway and other Companies, which are now very popular.

It is, I think, generally admitted that the ordinary stock and shares of trading and other Companies are not eligible for Assurance Societies' investments, as being too speculative. Yet so potent is the influence of the magic words, "Bank of England," that an exception is usually made in favour of Bank Stock, in apparent forgetfulness that the dividends of this great Corporation mainly depend upon the fluctuating and uncertain profits of a particular business—that of banking. How little the exception is justifiable by facts, a glance at the price lists of a comparatively short period will show. It will be found that, in 1840, the price of Bank Stock was marked at 156; while it is now (February, 1862), 244.

Leaving the consideration of what investments to avoid, for the more difficult task of what to select, it seems to me that, for what may be called the working capital, the smaller portion of the assets which is required to be readily convertible, the system of deposits with joint-stock banks and discount establishments affords the greatest advantages. These deposits are always available on demand, or at short notice, without expense, and without risk of depreciation like Exchequer Bills. I think that it is important that the amount to be thus held on deposit should be regulated always by the financial requirements of the Office, and never by the rate of interest. I have frequently had occasion to observe, that the effect of making transfers to permanent investments from deposit accounts, from dissatisfaction with the low rate of interest on the latter, has been to obtain a small increase in the interest at a serious cost to the capital. It will also probably be thought prudent to divide the total amount of deposits among different establishments.

According to the principles attempted to be laid down, the most desirable form of permanent investments seems to be bonds

and mortgages—including in that term every species of loan secured on tangible property; the preference being given to such as require policies as collateral security, as life interests, &c., where assurances are indispensable. I cannot think that it is altogether a justifiable proceeding to insist on assurances being effected which the security does not require, strong as may be the temptation to such a course; it would be better to stipulate at once for a higher rate of interest. Indeed, when a borrower finds the loan which he does require, subjected to a deduction for a premium on a policy of two or three times the amount, which he does not require—accompanied, as is sometimes the case, with a rather pharisaical announcement, that no higher rate of interest than 5 per cent. is ever taken—he must be reminded of the olden time, when the pecuniary necessities of gentlemen of the Charles Surface class were supplied in a currency composed of pictures, wine, and money, in very uncertain proportions.

As, however, the competition for such securities is on the increase, it is impracticable for Offices whose funds are large, to invest more than a part of their assets in this manner. In seeking for other descriptions of mortgages, our principles will lead us to select such as cannot readily be realised, avoiding those that are open to the competition of private trustees. For example, an advance under any of the land drainage Acts, in consideration of a rent charge for a term of years, will command a higher rate of interest than a mortgage in fee on the same estate, although the former charge has priority over the latter. In like manner, securities on the rates of counties and towns, and, generally, loans where the principal is repayable by instalments, are, I think, worthy of attention.

One other class of investments I will mention, which is quite in accordance with the principles laid down, and which seems to me peculiarly adapted for a portion of the funds of Life Assurance Societies—I mean the purchase of reversionary interests. A banker carries on a lucrative business by receiving money on deposit at one rate of interest and employing it at a higher rate; and I believe that a margin of 1 per cent., or less, will suffice to cover risks and expenses, and yield in addition a not inconsiderable profit. Surely, then, an Assurance Company may usefully employ, in the purchase of reversions, part of funds which are not repayable at short notice, but which are to accumulate during the lives of the assured, when it is considered that the margin is probably three times as great as suffices for the operations of the banker. I have,

therefore, never been able to understand how such institutions as Reversionary Interest Societies can have come into existence.

In concluding this hasty and imperfect sketch of a subject on which much difference of opinion exists, and on which a volume might be written, I have purposely avoided entering into details, or attempting to lay down rules by which the merits of any individual security are to be judged. I am aware, too, that other than merely financial considerations must sometimes be allowed their influence—as, for instance, it may be fairly urged that Government securities must be held to inspire confidence with the public. But, I think it will be admitted, that the finances of Life Assurance Societies ought to be managed on some clear general principles, and not, as is sometimes the case, left to depend on the passing impulses of the day, or on ideas derived from other and very different pursuits.

Memoir of the late John Finlaison, Esq., Actuary of the National Debt, Government Calculator, and President of the Institute of Actuaries.

FEW objects can be found more worthy of attention than the life of a man who has spent year after year in an unremitting and successful struggle against the ever-varying difficulties of new sciences; and who, not content with treading the beaten paths, has always sought to take a place among the pioneers of knowledge, neither deterred by the labour, nor dismayed by the risk of failure.

Mr. John Finlaison, late President of the Institute of Actuaries and Government Actuary, was eminently a man of this description. Born at Thurso, in Caithness, August, 1783, he lost his father when only seven years old. His innate energy of disposition, however, aided by the careful tutelage of his mother, procured for him that education for which Scotland affords such facilities; and he showed a keen taste for classical literature, and attained considerable local reputation as a poet. At this time his great mathematical talent was suspected by no one, least of all by himself, who regarded such studies with dislike; but the highly logical bent of his mind was shown in a faculty for systematizing, which he displayed to great advantage as factor to Sir Benjamin Dunbar, of Hempriggs, afterwards Lord Duffus, whose whole estates, with

those of Lord Caithness, were entrusted to his management at the age of 19.

The ability which he thus early displayed induced these kind friends to recommend that he should study for the Scottish bar. He accordingly went to Edinburgh, and qualified himself for that profession; but having visited London in 1804, on business, he became attached to the daughter of the Rev. James Glenn, and receiving from the then Secretary at War, the Hon. Mr. Dundas, afterwards second Lord Melville, the offer of an appointment under the Board of Naval Revision, which enabled him to marry at once, he was induced to relinquish the legal profession and enter the Government service in July, 1805, when Mr. Pitt was Prime Minister. He was shortly afterwards promoted to be first clerk to the Commission, and filled that office till the Board closed its labours in August, 1808. For some time previously he had also acted as secretary to a committee of the Board, represented by one of its members, Admiral Sir William Domett; and in that capacity, although but twenty-three years of age, he framed the eleventh and twelfth reports of the Commission—since laid before Parliament—and was the sole author of the important system therein recommended for the reform of the victualling departments. The accounts of this branch of the service had, for a century past, been so loose, imperfect, and complex, as to amount to a mere nullity; and, moreover, were seldom less than eighteen months in arrear. The effect of the new system was to produce them before the Board in London, simple, perfect, checked and audited, in *three weeks*. And this was accomplished not only without increased labour, but with a reduction of the establishment, which effected an enormous annual saving to the country. A memorial from the officers at Deptford stated, that, in the year when Mr. Finlaison's system first came into operation (the year 1809), the consequent saving from their yard alone was upwards of £60,000. Had the same system been established in the dockyards, many millions of the public money might have been saved.

On the dissolution of the Commission for the Revision of the Navy, the members of the Board expressed their great satisfaction with the ability evinced by Mr. Finlaison; and, in a letter to Mr. Wellesley Pole, then Secretary, they warmly recommended him to the notice of the Lords of the Admiralty, who were at that time much in need of the services of an able accountant. By the special advice of Sir William Domett, Mr. Finlaison was employed by Mr. Wellesley Pole to consider the state of the records and despatches

of the Admiralty; and, if possible, to devise some remedy for the inextricable confusion into which they had fallen.

Ever since the Restoration the records had accumulated without any means of reference, except from the accidental recollection of individuals; hence the production of the entire correspondence upon any one subject was almost impossible. Papers upon the most recent events, even when described and known to be in the office, could with very great difficulty be discovered. In 1805, Lord Barham had appointed an eminent barrister, Mr. Morthland, at a salary of £800 per annum, with two assistant clerks, to devise some remedy for the increasing evil; but, after eighteen months' trial, he completely failed, being deterred by the magnitude and difficulty of the object in view.

Mr. Finlaison undertook the task with great diffidence; he was young in official affairs, as in years; a stranger to the nature of the Admiralty correspondence, and unable to obtain from Mr. Wellesley Pole, who well knew the defect of the office, any idea of the way in which it was to be supplied. Under these discouraging circumstances—after a vain but most diligent search, in the public libraries and elsewhere, for any English or foreign system of assorting correspondence at all applicable to the subject—he could only promise that no personal labour should be spared to accomplish Mr. Wellesley Pole's wish. After nine months of incessant application, he produced a magnificent system of digesting and indexing the records and correspondence of the Admiralty, which, after the lapse of more than fifty years, still works with such perfection, that should information or precedent be needed on any subject under discussion, all that has been received concerning it can be immediately known; or should the name of some individual be called for, all that relates to him may be instantly produced. Under his superintendence, this system so eminently contributed to the dispatch of business, that every letter—although, in time of war, these averaged in number from two to three hundred a day—was answered on the very day of its arrival. The value of such precision and expedition in the conduct of naval affairs can hardly be appreciated by those who have not experienced the want of it. There was not, at that time, any system of official arrangement in any office of state in Europe which could be compared with this invention.

The following account of his work is extracted from a letter written by Mr. Finlaison in 1855, in answer to a request from the Admiralty that he would reconsider the working of his

system, and suggest some improvements which time had made necessary :—

"Among the manuscripts of Samuel Pepys, at Cambridge, will be found 'A Devout Imagination,' a sketch, in fact, of a return intended to show from time to time the nature and amount of the stores in hand in the several dockyards. This renowned Secretary of the Admiralty showed himself no mean adept in the art of arranging a system on the Linnæan principle, although he had died before the birth of its author. His kingdoms were—timber, the metals, fibril materials, and unctuous matter. His genera were the different kinds of each of those. His sub-genera, the raw and manufactured respectively. These last again were ramified into innumerable species and varieties, until you came to the individual article wanted—perhaps a common brass cock for a spirit cask.

"When I entered on my task at the Admiralty, I took the grand idea of Mr. Pepys for the basis of my plan. The Admiralty records were tolerably complete since the period of the Restoration; they consisted of letters and reports addressed to the Board, and of copies of letters and orders issued from it. I soon found that these papers were of three descriptions, and that it would be at once unprofitable and hopeless to attempt to digest the whole. There is, first, a vast correspondence relating purely to the interests of individuals; the discharge of impressed men; applications from officers for employment, for leave of absence, for particular information, &c., which I did not think worthy of record, beside subjects intimately connected with the public service. In the second place, a very large proportion even of papers regarding the public service are worthless as soon as answered; they consist merely of official reports and ephemeral details, such as periodical reports, reports of the arrival and sailing of ships, applications for supplies of stores, payment of ships, &c.

"The remaining portion of the records comprising the public acts of the Board is alone valuable for reference. A digest is made of these only, which form about a third of the whole correspondence; but the papers passed over can always be found, if needed, by means of the index, as will presently be seen.

"The purpose of any system of reference must be to supersede the use of individual memory—to instantly invest a new Board of Admiralty with the very same facilities for the despatch of business which had previously been at the command of their predecessors. The principle of my method is briefly this :—All information sought for must relate either to a person or a thing. The index, therefore, is an alphabetical list of proper names of persons and ships, in which the history of each name may be traced, though concerned in fifty different things; while the digest is a systematic list of words in which any subject may instantly be brought under review, though fifty persons have taken part in it. It was in the arrangement of this list that I found Mr. Pepys suggestion to be, like the poetry of Homer, a source of inspiration. One hundred and four chapters embraced all capital topics, such as firearms. I divided the chapters into sections as numerous as the different kinds of each of these, and placed in each section all discussions upon one and the same thing. The Minié rifle, for instance, ought to be found in one chapter and section of the digest, unencumbered by discussions on other sorts of gun-barrels or on firearms in general. In the one case a lord hears that something has happened—he knows not *what*

nor *when*—to a particular person or ship; he is instantly informed by means of the index. In the other case, a lord hears there are heavy complaints—he knows not *when* or *from whom*—of bad victuals. But, on reference to the digest, he is at once put in possession of the whole matter.

"In each of these books the utmost brevity is required. Immediately the letters reached me from the Board—generally the morning after they were received and acted on—I marked each, if of a memorable nature, in pencil, with the number of the chapter and section in the digest to which it belonged, as 51–1; having a sheet containing a plan of the digest as a guide. I also marked any proper name, as well as the signature, with a large cross ×. The letters then passed to the clerk of the index to be stamped, marked and numbered, for the different cabinets in which they were finally to be placed. This done, the clerk, whenever he saw a cross, without reading the letters, entered the name referred to in his index, together with the digest mark, 51–1, which referred to the subject-matter, and the numbers by which the letter itself might be found. It was estimated that in this way he made annually forty thousand entries. From him the letter went to the clerk of the digest, who opened the chapter and section referred to, and wrote therein an exact précis of the letter, and its answer already engrossed upon it by the lord whose department it concerned, together with the numbers referring to its place of deposit. If of an ephemeral nature, the letter was, of course, not marked for, and not entered in, the digest. The letters being indexed and digested, were, by the same two clerks, placed in their respective places. To a third clerk and myself fell the duty of producing any of the papers when wanted, and of compiling for the Board statements or memoirs on any topic whatsoever upon which information or precedent might be desired.

"To give the Board ocular proof, in the shape of an actual model, that there could be no doubt of the success of this plan, I took the whole correspondence of the year 1803, the first year of the war: assisted by one clerk, I made, in three months, an index of every name and ship mentioned therein. With my own hand I made a précis of all the *memorabilia* found in every single letter—and in that year these were not less than twenty-five thousand; placing experimentally in the margin some significant title. Revising these last carefully, generalisation—thus rising from analysis to synthethis—ensued, and my plan was developed and acted upon.

"A second division of clerks was appointed to digest and index, on the same principles, the records of former years since 1792, and thus the chain of naval history is complete from that period to the present time."

Baron Dupin, a member of the French Institute, gives a clear account of this system, while describing the Admiralty, in his work entitled *Voyages dans la Grande Bretagne.* He recommends it as a model for the use of his Government, which proves that it was highly esteemed in France. Louis XVIII., after his return to France, transmitted to Mr. Finlaison, in 1815, the order of the Fleur-de-lys—a gracious acknowledgment of services rendered him in connection with this subject. The invention was shown to other royal personages by Lord Melville, then First Lord of the Admiralty, who permitted them to carry away plans for the use of their

respective Governments. Among these were the Grand Dukes Constantine and Nicholas, the Archdukes John and Lewis of Austria, and the Grand Duke Michael of Russia, who visited the Admiralty at various times between the years 1816 and 1818; and to each of whom Mr. Finlaison presented an elaborate account of his method, and received testimonials of their satisfaction. He was honoured by the particular notice of the Grand Duke Nicholas, since Emperor of Russia, who sent him an autograph letter of thanks.

To further the efficient working of the system, Mr. Finlaison, at Mr. Wellesley Pole's strong recommendation, was, in 1809, appointed Keeper of the Records. A few months later, a library having been formed, consisting of about 5,000 volumes, he was made librarian, and became thus the general depositary of all the information, whether in print or manuscript, in the Admiralty. This necessarily entailed his being general referee for supplying that information to each member of the Board, to the public and private secretaries, and to every clerk in the office. Moreover, he became, in consequence, reporter and précis writer on all difficult and complicated inquiries arising from day to day. This highly confidential and important service—than which nothing can be more arduous—he fulfilled for twelve years. While he occupied this position, many members of the Board availed themselves of his freely-offered services, in matters foreign to his appointed duty and performed out of office hours. The following are a few instances in which his ability was remarkably displayed on important questions.

In 1810, Lord Mulgrave, then First Lord of the Admiralty, was called to account for his naval administration by Lord Melville, under the mask of a question on troop-ships. Lord Mulgrave, who had shown Mr. Finlaison much previous kindness, now requested him to prepare the materials for his defence in both Houses of Parliament. Mr. Finlaison accordingly, after three months of zealous and successful exertion, produced such an amount of information, calculation, and reasoning, as enabled Lord Mulgrave triumphantly to clear his conduct as a statesman, and obtained from him many expressions of gratitude and promises of protection, which, however, were not redeemed when needed in after years.

In 1811, by Mr. Barrow's desire, he compiled an account of the enemy's force, involving the analysis and comparison of more than eight hundred dispatches, as well as of many thousand foreign

and English newspapers. He formed a digest out of all this intelligence for every port and arsenal in Europe; and framed a list of every foreign ship of war, with particulars of their age, force, &c., and of the increase and loss of the enemy's naval power during the previous six years. This list experience showed to be perfectly correct, and it was even relied on in Parliament as accurate. Such information had never before been obtained with even tolerable authenticity.

In the same year Mr. Yorke employed him to investigate the abuses of the sixpenny revenue at Greenwich Hospital, which was a fund instituted in 1696, and intended to support the out-pensioners by a tax of 6*d.* a month on all seafaring men. He compiled, with great labour, a report on the subject, in which he showed that 10,000 additional pensioners might be constantly maintained on that establishment, and also that the allowance of the existing pensioners might be much increased by including the class of fishermen, boatmen, and mercantile sailors among the taxpayers, as well as by the reform of many abuses in the office and by the abolition of sinecure places.

For some years past petitions had been repeatedly sent up by the various departments of the Government, praying for a revision of their salaries. The matter having been more than twice forced upon the notice of Parliament, Mr. Finlaison was, in 1813, directed by Mr. Croker to inquire fully into the case of the Admiralty departments, and to apply systematic principles of justice to the consideration of the subject, which had never before been done. After six months of close attention, he completed the elaborate report upon which was founded the present system of salaries in the Admiralty.

In 1814 he compiled the first official Navy List, a work of immense though humble labour, the accuracy and usefulness of which is now well known. It was issued monthly, and as long as Mr. Finlaison remained at the Admiralty he undertook the troublesome duty of correcting and editing it.

From 1817 to 1818 he was occupied in framing one of the most useful records ever formed in any nation, a biographical register of every commissioned officer in the Navy—in number about 6,000—describing their services, merits and demerits. This laborious work he engrafted on to his system of the digest and index; and its value is obvious when it is considered that it affords the means of deciding, with perfect justice, on the claims of individual officers; and, in the event of any expedition

being undertaken, renders it easy to select the *élite* and flower of the Navy, or to fix at once upon those officers who, from previous local experience or other causes, would be most fitted for the service—a matter of great importance.

In 1819 his labours were still further increased by the resolution of the Lords of the Admiralty to apply his method of arranging papers to all the hydrographical information that had ever been received at the office, in manuscript, print, or charts, and to consign these last also to his custody. Deeply impressed with the importance of the measure, and fully aware of the immense benefit to hydrographical knowledge which should result from it, he undertook the task with energy and enthusiasm, though as the whole of his time was engrossed by the routine of his daily duties, the extent of geographical and nautical study which the nature of the service demanded, was entirely acquired in his few moments of leisure.

Mr. Finlaison had thus, during the ten years of his official career, accomplished no fewer than ten important and laborious public services, of which some were in their nature ephemeral, though of the utmost consequence at that time, while others—for instance, the Admiralty system and the Navy List—having stood the assaults of adverse criticism, and the more formidable tests of time and of two great wars, continue unchanged in design to the present day. He also introduced into the Record Office, and strictly enforced, a custom of civility towards the public and a readiness to impart information, which, had it been universally carried out, would effectually have prevented the "Circumlocution Office" from acquiring celebrity.

At this time the reputation which he had gained induced Mr. Barrow, and several other distinguished members of the Royal Society, to recommend that he should become a candidate for election into that body. He, however, declined to do so, from motives which had their origin in the peculiar independence of his disposition. He preferred that his name should be known to scientific men from his works, and not from his being casually thrown among them. He undertook, also, about this period, to frame a digest of the library of the Royal Society upon the plan of the Admiralty system. An outline of this plan was laid before the council and approved, but the immense call upon his time in matters of state, in 1819, compelled him reluctantly to abandon the work.

The position of librarian at the Admiralty having brought

before his notice many valuable State papers relating to the American rebellion, he was, in 1813, induced to attempt the completion of Sir Redhead Yorke's Naval History, which was intended to form part of Campbell's *Lives of the Admirals*. Mr. Finlaison carried out his design in part, by continuing the History down to the year 1780. His remarks on the political aspect of the American question show both a very wide and accurate extent of reading, and also a knowledge of the true principles of commercial and colonial policy, very unusual at the time when he wrote; at the same time his analysis of Rodney's inglorious action with De Guichen, off Martinique, on the 17th of April, 1780, shows an acquaintance with naval strategy which gained him very high commendation from Sir W. Domett and other eminent naval officers. This portion of the work was printed for private circulation, but, for many reasons, its publication was delayed; and afterwards, when his attention was turned to the subject of life assurance, he found himself unable to spare the time necessary for completing the history, and finally abandoned its further progress.

In 1815, Dr. Barry O'Meara, having been appointed physician to Napoleon at St. Helena, commenced a correspondence with Mr. Finlaison, his private friend, on the subject of the Emperor's daily life. Copies of these celebrated letters were given by him to Mr. Croker, and by him transmitted to the Government and the Prince Regent. Dr. O'Meara wrote under the positive assurance that these letters should never be suffered to become public; and, in 1824, Mr. Finlaison burned them all by his desire. Some of the copies, however, fell into other hands, and were published in 1853, in a book entitled *Napoleon at St. Helena, and Sir Hudson Lowe*. Intelligence of the proposed publication of this work having reached the ears of Lord Clarendon, he at first thought of suppressing it by an injunction; but before instituting proceedings, he requested Mr. Finlaison to read over the proof sheets and give his opinion as to the danger of permitting its issue. Having, with some difficulty, obtained the proof sheets from the publisher, Mr. Finlaison found that the author had not obtained possession of any documents of importance; and Lord Clarendon, judging that it was not a book likely to attract much attention, withdrew his opposition.

He now completed a work upon which he had been zealously employed since 1812. This was the fund for the maintenance of the widows and orphans of all who were employed in the civil departments of the Royal Navy. He first suggested it in a letter

to the heads of the departments, and at once obtained Mr. Croker's full co-operation and warmest approbation. Mr. Finlaison then first directed his attention to the principles of life assurance, a science which was little understood. It was upon this study that his future labours were to be spent, and on the investigation and discovery of its laws that his eminent reputation was to depend—a reputation that gained him honourable notice in the history of his country.*

Starting with small knowledge, he worked assiduously and ably for seven years, during which time he mastered the principles of the subject, constructed the necessary tables, matured the whole plan of the undertaking, and obtained the means of forming a Society to execute it. Through Lord Melville's intervention, his generous efforts terminated successfully, by the establishment of the fund by Order in Council, 17th September, 1819.

Many other Societies were framed on this model: among others, the Naval Medical Supplemental Fund, for providing an annual pension, of not less than £40 additional to that allowed by Government, to the widows of medical officers. This charity, although commenced subsequently to the preceding, and in imitation of it, was completed before it, and established under Government authority in August, 1817. Mr. Finlaison took the greatest interest in this Society, which owed its existence and subsequent prosperity entirely to himself. His financial skill enabled him to construct its valuable system of accounts, which he gradually perfected by the experience of several years; and his management as secretary, gave the Society, in the twelfth year of its existence, an annual income of £9,000, with trifling expense to individual members. The unworthy and ungenerous treatment he then received from its directors, who not only refused to remunerate him for a new and elaborate system of tables, but endeavoured to injure his character in the eyes of the Chancellor of the Exchequer, by the grossest falsehoods, compelled him to throw up the ungrateful office of secretary and manager, after he had, during twelve years, devoted much of his time and talents, almost gratuitously, to the service of the Society. In a letter written at the time of his resignation, he speaks thus of the stability of his scheme:—

"Many similar Societies have been dissolved through their unskilful construction, while this institution, to which I can triumphantly refer as the

* *Vide* Lord Macaulay's *History*, vol. i., p. 283.

only one in the empire which has its foundation deeply laid in scientific skill, is established on such principles of cohesion, the result of long and patient investigation, as that neither time nor accident, except the suicidal act of its own rulers, can in the slightest degree affect its efficacy."

It is worth recording that this "suicidal act" did take place; and that, owing to the mismanagement of its directors, the Society gradually sank, till, singular to say, it died in the same year with its founder and benefactor.

The success of the charity thus founded by Mr. Finlaison, together with his subsequent services in an investigation into the general condition of Friendly Societies, upon which he was employed by a Select Committee of the House in 1824, naturally gave rise to a private practice among the various Benefit Societies which speedily sprang up in all parts of England. He constructed tables for many of these, furnished the scheme of some, and entirely constituted others.

From the beginning of the year 1819, Mr. Finlaison, though he still retained the office and performed the duties of Keeper of the Admiralty Records, found himself required to devote all the time and skill at his command to the service of the Treasury, which then stood in great need of an able financier. During the previous year, arrangements had come under the consideration of Government for the payment of all half-pay and pensions by some means beyond the current service. Lord Melville was well aware of Mr. Finlaison's value as a trustworthy public servant, and he also knew that he had acquired considerable reputation at the Admiralty by his knowledge of the principles of finance. He, therefore, advised the Chancellor of the Exchequer (Mr. Vansittart, afterwards Lord Bexley) to entrust to Mr. Finlaison the very confidential service of maturing a scheme for meeting this expense. This was to raise a fund, by resuscitating an old corporation, so as to be in some sort a rival to the Bank of England. In the year 1808 two remarkable modifications had been introduced into the plan for the redemption of the National Debt. The first, brought in by Mr. Percival, was the great operation of granting life annuities chargeable on the Sinking Fund; the other was the savings banks system, soon afterwards commenced by Mr. Rose. Upon both these schemes Mr. Finlaison was now desired to comment, in connection with the "dead weight," by which name the half-pay and pension expenditure was familiarly known, from the accident of Lord Castlereagh so

calling it in the course of a debate. We shall say nothing at present of the report on the savings banks, which led to the commutation of the half-pay and pensions, nor of Mr. Finlaison's services in connection with this measure. They were subordinate to the work suggested by his second report, and to which, for nearly ten years, every faculty of his mind was devoted. His report on the life annuity system was laid before Mr. Vansittart on the 1st September, 1819; and the learning, ability, and extensive acquaintance with every bearing of the subject which it displayed, proved it to be the fruit of many years' previous study. In it Mr. Finlaison first directed the attention of the Chancellor of the Exchequer to the ruinous loss sustained by the Government in granting life annuities to the public at prices immensely below their value. This loss, which had already, in eleven years, amounted to nearly two millions, began from the circumstance that the Treasury had no actuary attached to its service. In the year when the life annuity system was instituted, 1808, there was but one eminently scientific man in practice as an actuary in all England, Mr. W. Morgan, of the Equitable Life Assurance Office; and there was but one English observation on the mortality of mankind—viz., that formed on the bills of mortality in Northamptonshire by Dr. Price, the uncle of Mr. Morgan, expressly for the benefit of the Equitable. How beneficial it was had appeared from Mr. Morgan's Address of the 24th April, 1800, in which he published a statement of the experience of the Equitable during the previous thirty years, which showed that the real decrement of life under the age of thirty was *one-half* that indicated by their tables; at other and higher ages the difference was *two-thirds* and more. Their gains were accordingly enormous. It is obvious that the tables which were so gainful to the Equitable, who, in the shape of premiums, *purchased* life annuities, must be ruinous to the public revenue which was to *sell* them. Nevertheless, the Treasury, in profound ignorance of this palpable truth, applied to Mr. Morgan to furnish tables for the newly-instituted Government annuities; who, accordingly, computed them on the basis of those very Northampton tables he had just proved so inaccurate; and the Government not only accepted his advice, but persisted in that erroneous measure of value for twenty years. Mr. Finlaison's report was not printed till 1824, when it was laid before the committee of finance; but Mr. Vansittart's confidence in the correctness of the Northampton tables was greatly shaken by the revelations it con-

tained, and he at once directed Mr. Finlaison to spare no labour in the investigation of the true law of mortality prevailing in England among the class of annuitants. He was thus plunged into the abyss of a new science, and in the watches of many a night slowly but surely traced out the fundamental laws of life.

He attempted at first to procure the necessary data from the records of the births, marriages, and deaths of the nobility of Great Britain; but, after consuming eighteen months in the trial, he found it would involve too wide a range of reading for his immediate purpose; and, although he had made considerable progress in this laborious research, he was obliged reluctantly to suspend it, and to turn to the Exchequer offices of England and Ireland. Here he found the very materials for his purpose in the records of the various tontines from 1695 to the great tontine in 1789, which, together with the annuitants of the Sinking Fund, and an observation made by himself on the Chelsea and Greenwich pensioners, afforded him the means of comparing more than 19,000 deaths. The incalculable advantage that these data afforded him arose from the fact, that in every instance the age of the deceased was delivered on oath; and, in the case of tontines, that of the surviving co-partners also. In another respect these observations had possibly an advantage, namely, that being taken only from the higher classes, they excluded those lives which could not have any concern with life assurance.

The only observation that had hitherto been made in England, in addition to the old Northampton tables, was the work of his contemporary, Mr. Milne, of the Sun Fire Office, whose data were only 900 deaths of both sexes indiscriminately, and whose ages were most imperfectly conjectured. Some idea may be formed from this fact of the superior extent of Mr. Finlaison's researches, and of the means he had of attaining perfection in his results. But the gradual conviction forced upon Mr. Finlaison, that none of his predecessors in political arithmetic had ever attained correctness, obliged him to repeat his work twice, thrice, even four times, before the truth of the new results became unquestionable; and the time which, during four years, he devoted to the intense labour of calculation, very considerably exceeded an average of *twelve* hours a day. With less enthusiasm in the cause, his task would not have been concluded in any reasonable period.

The methods of adjustment which he employed for this purpose are given in his report, 1829. From this it appears that

nineteen observations were adjusted from the formula (Milne's notation)—

$${}_1a=\sqrt[3]{\sqrt[5]{{}_1a_3\,.\,{}_1a_2\,.\,{}_1a_1\,.\,{}_1a\,.\,{}^1_1a}\times\sqrt[5]{{}_1a_2\,.\,{}_1a_1\,.\,{}_1a\,.\,{}^1_1a\,.\,{}^2_1a}\times\sqrt[5]{{}_1a_1\,.\,{}_1a\,.\,{}^1_1a\,.\,{}^2_1a\,.\,{}^3_1a}}\,;$$

while two more, being those from which the tables of life annuities were calculated, were adjusted from the formula—

$${}_1a=\frac{1}{25}(5\,{}_1a+4\,{}_1a_1+4\,{}^1_1a+3\,{}_1a_2+3\,{}^2_1a+2\,{}_1a_3+2\,{}^3_1a+{}_1a_4+{}^4_1a).$$

The result of his studies was the discovery that the average duration of human life had sensibly increased during the century. These tables also were the first which exhibited the difference between male and female mortality, although it had been loosely remarked for many years previously. Their accuracy in this respect has been impugned, and, in the case of advanced ages, appears to require some modification; but the recent elaborate and very extensive researches made by his son and successor, Mr. Alexander Finlaison, have shown that the error is small, and that he had attained almost as great accuracy as was possible with the data at his disposal.

The following extracts from the reports of 1829 and 1860 will exhibit the difference in a few cases.

The results in Mr. John Finlaison's report are extracted from observations 9 on female life, and 20 on male life, pages 44 and 58. Those in Mr. Alexander Glen Finlaison's are taken from observations 14 and 15, pages 140 and 144.

Males.

Age.	1829. Mortality per Cent.	1860. Mortality per Cent.	Difference.
3	1·24862	0·97842	·27020
8	·60403	·64628	–·04225
13	·52586	·59678	–·07092
18	·93132	1·04714	–·11582
23	1·41513	1·36790	·04723
28	1·27163	1·17380	·09783
33	1·24605	1·17550	·07055
38	1·31414	1·32440	–·01026
43	1·35342	1·36200	–·00858
48	1·42012	1·54340	–·12328
53	2·14940	2·08640	·06340
58	2·96085	2·59970	·36115
63	3·55167	3·45860	·09307
68	5·26964	5·24080	·02884
73	7·22627	7·53160	–·30533
78	9·79843	10·99010	–1·19167
83	16·22553	16·23580	–·01027
88	30·95338	21·95490	8·99848
93	46·00000	31·91890	14·08110
98	100·00000	71·42857	28·57143

Females.

Age.	1829. Mortality per Cent.	1860. Mortality per Cent.	Difference.
3	1·09779	1·07493	·02286
8	·57219	·61279	– ·04060
13	·47732	·61345	– ·13613
18	·83950	·82793	·01157
23	·86238	·82857	·03381
28	·90490	·86084	·04406
33	·99023	·95716	·03307
38	1·11019	1·02467	·08552
43	1·15152	1·10770	·04382
48	1·19281	1·27960	– ·08679
53	1·28756	1·51620	– ·22864
58	1·69708	1·87270	– ·17562
63	2·27646	2·62970	– ·35324
68	3·39883	4·06110	– ·66227
73	5·47601	6·07950	– ·60349
78	8·75801	9·42750	– ·66949
83	11·38146	14·45910	– 3·07764
88	16·05611	20·67350	– 4·61739
93	34·81818	29·32050	5·49768
98	65·00000	71·42857	– 5·42857

By the end of the year 1823, Mr. Finlaison had completed a set of tables on single lives, constructed on the basis of his recent discoveries, and adapted to the purposes of the Sinking Fund. In these he gave the value of immediate and deferred annuities at nine rates of interest, payable half-yearly, for every year and half year of human existence; and it may be observed, that no table of half-yearly annuities had till now been completed in this or any other country. He afterwards calculated a similar system of tables for two joint lives.

The Institute of France took considerable interest in the inquiry upon which Mr. Finlaison was engaged. The Academy of Sciences transmitted many questions on the subject for Mr. Finlaison's consideration; and, in return for the information contained in his answers, they did him the honour to send him a letter of thanks.

The savings bank system, and the report which Mr. Finlaison was desired to make on it, have already been mentioned as contemporary with the report on the annuity system, which led to such important results. His services in connection with the former subject may now be briefly enumerated in the order of their occurrence: they were performed during the progress of his calculations for the tables of life annuities, at the expense of intense mental toil. Before the close of 1819, Mr. Finlaison had furnished the Chancellor of the Exchequer with a very elaborate statement of the

age of each individual in the receipt of naval half-pay or pensions —14,000 persons, thence deducing the decrement of life among them. In 1821 Mr. Harrison employed him for several months in computations relative to the Superannuation Act, and expressed himself much pleased with the exertions and skill displayed by Mr. Finlaison; and in 1822 he was occupied in considerations relative to the commutation of the naval and military half-pay and pensions—which measure, suggested by him in his report on the savings banks, was finally established by the negotiations with the Bank of England, in 1823, for its acceptance of the charge for public pensions in consideration of the "dead weight" annuity. All the calculations connected with this subject were performed by Mr. Finlaison with such unsparing exertion and arithmetical skill as should neither be forgotten nor undervalued—especially as it was plainly stated in the House of Commons, that, in the whole establishment of the Bank, there was not one person capable of computing the new annuity at the fractional rate of interest agreed upon. This measure having passed into a law, it became necessary to ascertain the real decrement of life among the pensioned and half-pay officers of the Army and Navy. The result of Mr. Finlaison's investigations presented a most perfect view of the extent and progressive diminution of this department of the national engagements, similar to the view of the *rentes viagères* annually published by the French Government. Mr. Finlaison had previously completed the observation on the decrement of life among the Greenwich and Chelsea pensioners, already mentioned as forming part of the data for his law of mortality. It was formed from a registry of more than 70,000 persons, including 8,000 deaths, and presented important matter for the consideration of the Chancellor of the Exchequer.

The business of the savings banks was now committed to the National Debt Office, and an actuary and check-officer was needed. The Government, therefore, resolved to remove Mr. Finlaison from the Admiralty, to fill the newly-created office in the Treasury; and, on the 1st January, 1822, he was appointed Actuary and Principal Accountant of the Check Department of the National Debt Office, the duties of which position he performed for twenty-nine years.

From the first moment of his undertaking the subject of life assurance, until the production of the report of 1829, he continued constantly impressing upon the members of the Government the enormous loss which the country sustained by persisting in using the erroneous tables; but, so far from meeting with the encourage-

ment and commendation which might have been expected, he received nothing but neglect and contempt. Persons in high position refused to see him; and, when he had positively forced himself into their presence, would give no heed to his statements. And yet, when the matter came before Parliament, these were the very people who publicly declared that "they could never get any account from him!"

The anxiety which he suffered during the time when he was devoting his best energies to the service of his country, may be traced in his letters. He writes to Mr. Walpole, Secretary to the Chancellor of the Exchequer, in 1824, of the extreme cruelty "of leaving me to expend my talents, time, and labour, like a galley slave, at the rate of sixteen hours a day, eleven of which were constantly employed on the calculations for the life annuities; and this without being able, by the most fervent eloquence or the most pathetic complaints, to extract one shilling from the Treasury, as the records of that office will show, between Lady Day, 1819, and the month of April, 1821, save the sum of £500; and out of these, my private resources, I had to pay the clerks employed, to find them accommodation, to find an extensive collection of books, to defray the expense of stationery and all other contingencies that might arise."

Again, in a letter to a noble lord, dated March, 1828, praying for his assistance to defend his reputation in the House of Commons, when the correctness of a recent report by him, on the tremendous loss the Government was still sustaining in the matter of life annuities, was impugned by the ministry, he writes:—"In the simplicity of my heart I fancied that a service which went to prevent the loss of millions of the public money would have been rewarded by honour and profit. I have been fatally undeceived: my salary has been stopped; I have met with nothing but envy, malice, and hatred, for my discovery: and, although I have every year reiterated my warnings, not a jot of attention was paid me till now, when my statements are called extravagant. This is the same imputation on an *actuary's* character as cowardice is on that of an *officer*, and cannot be endured."

It was the accidental production of a letter from Mr. Finlaison, addressed to the Secretary of the Treasury, Mr. Herries, and dated 30th April, 1827, that occasioned the inquiry alluded to above. It was inadvertently laid before Lord Althorp's committee of finance by the Treasury themselves, in March, 1828. On the motion of Earl Grey, this letter was printed by order of the House of Lords,

and it was then shown that the annuitants of the first year, ending September, 1809, had already received back principal and interest of all that they had paid, and ought, therefore, in justice, to be all dead. But, far from being so, four out of every ten were still alive, and likely to live for many years. The same letter established, that, in the previous year, 1827, the loss sustained by the public revenue was advancing at the rate of £8,000 PER WEEK! and it further proved that this loss was concealed by the method of preparing the yearly accounts. This discovery caused the immediate suspension of the life annuity system, which was remodelled upon the basis of Mr. Finlaison's tables, and resumed in November, 1829.

In the same year Mr. Finlaison produced his report on the evidence and elementary facts on which his new tables of life annuities were founded, printed by order of the House of Commons, September, 1829. This important parliamentary document contained twenty-one new observations of the law of mortality, and one of the law of sickness prevailing among the labouring classes in London.

In the recent report on life annuities by Mr. Alexander Glen Finlaison, it is shown "that the loss saved to the Government by the adoption of Mr. Finlaison's tables, was, in five years, £390,501; by which it may be estimated that the substitution of Mr. Finlaison's tables for the old Northampton rates has already made a difference to the country of nearly £3,000,000 sterling."

In the year 1830 Mr. Finlaison was occupied in the investigation of the affairs of the London Life Assurance Association, and drew up a report containing many valuable suggestions for their benefit. By the end of the year 1831, he had, in conjunction with Professor De Morgan, completed an elaborate report on the affairs of the "Amicable"; and, in the succeeding year, he furnished them with a new set of tables. He also computed the tables for the Royal Naval and Military Life Assurance Society, which was established in 1837. Mr. Finlaison had a large share in the constitution of this Society, where he held, till his death, the office of consulting actuary. It may be mentioned here also, that, in 1839, he furnished the scheme and prepared the tables of the New York Life Assurance and Trust Company, the first institution of the kind established in the United States.

As the Government adviser on all measures involving political arithmetic, Mr. Finlaison's counsel and powers of calculation were

in constant requisition; and, from the year 1830 till his retirement from official life in 1851, there were few sessions of Parliament in which he was not called as a witness before some committee for the consideration of some important financial measure; few Crown commissions to which he did not render some important service—of which that in 1849, on church leases, under Lord Harrowby, may be specially mentioned; and scarcely a single department of the Government which did not apply to him for advice and assistance.

In 1833, by direction of Viscount Goderich, he commenced extensive computations of the duration of slave and creole life, with reference to the emancipation of the slaves on the West Indian plantations. In 1835 Mr. Finlaison performed the calculations necessary to effect the West India loan of £15,000,000, for compensating the slave-owners; and, in answer to certain resolutions on that loan, moved by the late Mr. Hume on the last day of the session of that year, he prepared a report early in 1836, which, as a Parliamentary paper on the higher finance and the funding system, is well worthy of perusal. Mr. Hume's reputation for financial knowledge was so completely annihilated by it, that he was never afterwards regarded as an authority on matters of finance, Mr. Finlaison having proved him to be completely in error on thirteen important questions. His masterly treatment of the subject did not fail to be duly appreciated by Lord Monteagle, by whom the loan was effected.

The demands made on Mr. Finlaison's mental powers had been so great throughout the progress of the measure for the abolition of slavery, that, at its conclusion, he became seriously ill. Great fears were entertained for the safety of his over-wrought mind, and he obtained permission from Lord Monteagle to reside in the country, and thenceforward was obliged to exercise more caution in his devotion to the public service. About this time, being a widower, he married the daughter of Mr. Thomas Davis, of Waltham Abbey.

His valued friend, Dr. Southwood Smith, the father of sanitary reform, having submitted to him the experience of the London Fever Hospital, of which he was physician, for the ten years preceding 1834—an observation which included nearly 6,000 patients—Mr. Finlaison was enabled to make a calculation of the mortality of fever, which presented some curious and instructive results. These were published, in 1835, by Dr. Southwood Smith, in a work entitled the *Philosophy of Health.*

His professional researches were still assiduously carried on, and his knowledge and counsel were called for, to an extent greatly beyond what is generally known, in the measures emanating from the Ecclesiastical Commissioners. In proof of this, his numerous reports may be adduced, on the means of improving church property by the abolition of fines, in 1835; on the great question of church leases, and the steps leading to the "Appropriation Clause," in 1836; and on the church-rate question, in 1837.

There was one service in connection with these measures to which Mr. Finlaison was wont to refer, with some consciousness of superior talent, as one of pre-eminent merit. Some time before the Parliamentary contest on the subject of church rates, Mr. Finlaison, in November, 1836, attended the Cabinet, and had the honour of explaining to the assembled Ministers, that he could demonstrate, by the mere force of arithmetic, and without any survey, the rental of the episcopal and capitular estates, at a minimum. Earl Russell, who was present, might still possibly remember how incredulous many members of the Cabinet were, and that Lord Sidmouth pronounced such a thing to be a flat impossibility. But Mr. Finlaison did prove it, in two memorandums to Lord John Russell, dated January, 1837, which were found to be unassailable. His opinion was fully relied on, and this most confidential service was matured in secresy, until the month of April, 1838, when his reports were laid before Parliament.

Mr. Finlaison was consulted on certain points connected with the establishment, in 1837, of the registration of births, deaths, and marriages; and the closeness of his estimate of the deaths to be registered the first year—falling within twelve of the whole number recorded, 334,000—attracted much notice when mentioned in the Registrar-General's First Annual Report. He also predicted the number which would be found at the ensuing Census of 1841, with the object of enabling the Registrar-General to estimate the number of clerks to be provided, and he came within 1,200 of the whole female population of Great Britain.

The last Committee of the House of Commons before which he attended as a witness was that which sat, in 1848, upon Feargus O'Connor's Land Scheme. In the same year he sent in to the Treasury the second of two reports on the Act of the previous session for lending money to the Irish landlords, with the tables necessary to give effect to it. None but a very able actuary

could have computed them, from the complexity of some of the conditions.

In 1847, the Institute of Actuaries having been formed, Mr. Finlaison was unanimously elected President, and filled that post till the day of his death. It is not necessary to enlarge upon the affectionate regard which he entertained for all the members, nor upon the uniform kindness which they exhibited towards him. These are best shown by the letters which passed between them in 1851, when, being attacked by a dangerous, and, as it was thought, incurable disease, he tendered his resignation.

"*To the Vice-Presidents and Members of the Council.*

"26*th April*, 1851.

"My dear Sirs,—Having, for many months past, been afflicted with a disease, which, in all probability, will soon be mortal, I feel that I should make a bad return to the Institute for the honour they have so repeatedly conferred on me, in choosing me their President, if I now continued to hold that distinguished office; and especially in such a time as the present, when I am totally unable to acquit myself of its duties.

"I request you, therefore, to tender my respectful resignation of the office of President, with my grateful acknowledgments for the kindness and deference which, on every occasion, I have received at the hands of its members.

"To yourselves, my dear Sirs, collectively and individually, without exception, I never can find words to express the sense I entertain of your kindness to me whenever and wherever we have met; and let it not repent you that, however unmerited on my part, you have shown me such kindness, when I inform you that I receive it as a testimonial from the *élite* of my brethren that my life has hitherto not been unprofitably spent, but that I have, in some degree, enlarged the boundaries of a science that, even in its infancy, promises to be of the utmost benefit to mankind.

"Farewell.

"Your affectionate brother and servant,

"John Finlaison."

"John Finlaison, *Esq., President of the Institute of Actuaries.*

"3*rd May*, 1851.

"Dear Mr. President,—We are deputed by the Council to acknowledge the receipt of your letter of the 26th ult., and to assure you that its contents have been the occasion of very sincere regret to every member. Although aware that you had been for some time suffering from indisposition, the Council generally were not in the least conscious that your malady was of so severe or so alarming a character, and they have received the information with deep and unfeigned sorrow.

"We are requested to convey to you warm and earnest hopes that your valuable life may be spared for many years to come. We are also to convey to you the urgent request of the Council that you will continue to

retain the Presidency of the Institute, and remain the head of that profession which you have been in so great a measure the means of creating, and for which you have so zealously and faithfully laboured throughout your useful life.

"Believe us, dear Sir,

"With much respect,

"Your faithful servants,

"Jenkin Jones,
"Robert Tucker, } *Hon. Secs.*"

In August, 1851, Mr. Finlaison finally retired from his position as Actuary of the National Debt and Government Calculator, after a confidential and devoted service under the Crown of nearly half a century, forty-two years of which were passed as chief clerk of one or other of two most important public departments—a service unalloyed by the slightest censure.

The last nine years of his life were passed in comparative ease and tranquillity. He applied himself, with unimpaired energies, to his favourite studies of Scriptural chronology and the universal relationship of ancient and modern weights and measures. His researches, which were exceedingly profound on both subjects, led him to form opinions decidedly adverse to the introduction of the decimal system of coinage and metrology into this country. He was on the point of completing a work in which his knowledge of numerical relations gave him unusual advantages, and, in combination with his native mental abilities, promised to render such a treatise of the utmost value to the student of biblical literature, when he was unexpectedly attacked by congestion of the lungs, and, after a brief illness of one week's duration, he passed away, on the 13th April, 1860, in his seventy-seventh year, from a scene in which he had long taken no useless part.

The reader of the foregoing outline of the chief events of Mr. Finlaison's active life, can hardly fail to be struck with the energy and unceasing industry which enabled him to accomplish so many entirely different undertakings. That an orphan boy should raise himself, by his own industry, to acquire a European celebrity, is a matter for high praise; but it is more surprising that a man, whose taste and studies had been entirely devoted to literature, should, by mere perseverance, gain a mastery over an abstruse science to which he had a positive dislike at starting.

The poetical talent which he possessed as a boy, he cultivated with great success, and produced some lyrical poems of considerable merit. He was also an enthusiastic mason, and attained a

high rank in that mysterious fraternity. He bore the chivalrous character of a past age. Incapable of acting with meanness, he could hardly be brought to believe it in others; and he suffered through life from the implicit confidence which he placed in every one. Injustice or oppression he would never endure; and for the oppressed he has not only afforded his time and energy, but has risked making powerful enemies, and on one occasion hazarded his whole fortune on behalf of a perfect stranger. He was affectionate in disposition, witty and animated in conversation, cool in moments of danger; and he died as he had lived, with the courage and resignation of a Christian gentleman.

On the Construction and Use of Commutation Tables for Calculating the Values of Benefits depending on Life Contingencies. By PETER GRAY, *Esq.*, F.R.A.S.*

(Continued from page 104.)

PROBLEM VI.—To find the value of a life annuity on (x), of which the first payment is to be £1; the second, £2; the third, £3; and so on, increasing annually, by the amount of the first payment, to the end of life.

This benefit may be conceived to consist of a series of annuities of £1, of which the first is to be entered upon now; the second, one year hence; the third, two years hence; and so on, an additional annuity being entered upon every year, till the end of life. The present values of these annuities are, by Problems II. and III.,

$$\frac{N_x}{D_x}, \quad \frac{N_{x+1}}{D_x}, \quad \frac{N_{x+2}}{D_x}, \quad \text{\&c.};$$

and the sum of these expressions, which will be the required present value, is, by (2),

$$\frac{N_x + N_{x+1} + N_{x+2} + \text{\&c.}}{D_x} = \frac{S_x}{D_x}.$$

The same result will be obtained by viewing the benefit in another light. Thus, the first payment is £1, to be received a year hence; its present value, therefore, is $\frac{D_{x+1}}{D_x}$. The second payment is £2, to be received two years hence; its present value, therefore,

* Extracted from the *Mechanics' Magazine* for 1842.

is $\frac{2D_{x+2}}{D_x}$. In like manner, the present values of the third, fourth, &c., payments will be respectively

$$\frac{3D_{x+3}}{D_x},\ \frac{4D_{x+4}}{D_x},\ \text{\&c.}$$

and the sum of these expressions, which will be the present value of the benefit, is by (11), $\frac{S_x}{D_x}$, as before.

Example.—Required the present value of an increasing annuity of £1, £2, £3, &c., on (30).

$$\textit{Answer.}\ \frac{S_{30}}{D_{30}}=\frac{539239{\cdot}0044}{2257{\cdot}6521}=238{\cdot}8494=\pounds 238.\ 17s.$$

PROBLEM VII.—To find the present value of a deferred increasing annuity on (x), that is, of an annuity to be entered upon n years hence, and whose first payment is to be £1; the second, £2; the third, £3; and so on, increasing every year, by the amount of the first payment, to the end of life.

This benefit may be considered as a series of deferred annuities of £1, of which the first is to be entered upon n years hence; the second, $n+1$ years hence; the third, $n+2$ years hence; and so on, an additional annuity being entered upon every year, till the end of life. The present values of these annuities are, by Problem III.,

$$\frac{N_{x+n}}{D_x},\ \frac{N_{x+n+1}}{D_x},\ \frac{N_{x+n+2}}{D_x},$$

and so on; and the sum of these expressions, which will be the present value required, will be, by (4), $\frac{S_{x+n}}{D_x}$.

Example.—An annuity, whose successive payments are to be £1, £2, £3, &c., is to be entered upon by (20), 10 years hence, that is, when he attains the age of 30. Required its present value.

$$\textit{Answer.}\ \frac{S_{30}}{D_{20}}=\frac{539239{\cdot}0044}{3818{\cdot}9594}=141{\cdot}2005=\pounds 141.\ 4s.$$

PROBLEM VIII.—To find the present value of a temporary increasing annuity for n years on (x), the first payment being £1; the second, £2; and so on, increasing each year by the amount of the first payment, till the end of the term, when all payments cease.

This benefit consists of a series of temporary annuities of £1, of which the first is to be entered upon now, to continue n years, and then to cease; the second is to be entered upon one year hence, to continue $n-1$ years, and then to cease; and so on to the

nth annuity, which will be entered upon $n-1$ years hence, continue one year, and then cease—that is, will make but a single payment.

Now, the present values of these annuities are (omitting, for brevity, the common divisor D_x), by Problems IV. and V.,

$$\begin{array}{rl} \text{Of the 1st,} & N_x - N_{x+n} \\ \text{2nd,} & N_{x+1} - N_{x+n} \\ \text{3rd,} & N_{x+2} - N_{x+n} \\ \cdots & \cdots\cdots \\ \text{Of the } n\text{th,} & N_{x+n-1} - N_{x+n}. \end{array}$$

The sum of these is, (7),

$$S_x - S_{x+n} - n\,N_{x+n};$$

and, inserting the divisor, we have for the required present value,

$$\frac{S_x - S_{x+n} - n\,N_{x+n}}{D_x}.$$

Example.—Required the present value of an increasing annuity of £1, £2, £3, &c., on (55) for the next 10 years.

Answer. $\dfrac{S_{55} - S_{65} - 10\,N_{65}}{D_{55}} = \dfrac{60976{\cdot}4989 - (17008{\cdot}1044 + 23769{\cdot}292)}{587{\cdot}3514}$

$$= \frac{20199{\cdot}1025}{587{\cdot}3514} = 34{\cdot}390 = £34.\ 7s.\ 10d.$$

Problem IX.—To find the present value of an arrested increasing annuity on (x), that is, of an annuity whose successive payments are to be £1, £2, £3, and so on, increasing £1 every year till n payments have been made, the nth payment being thus £n, which is also to be the amount of each subsequent payment till the end of life.

The difference between this benefit and that which formed the subject of the last problem is, that while, in the case of the benefit referred to, all payments were to cease at the end of n years, in the present case the increase only is arrested, and the annuity continues to pay £n a year during the remainder of life. If, therefore, to the present value of the benefit in the last problem we add the present value of an annuity of £n, deferred for n years, the sum will be the present value required. That is—

$$\frac{S_x - S_{x+n} - n\,N_{x+n}}{D_x} + \frac{n\,N_{x+n}}{D_x} = \frac{S_x - S_{x+n}}{D_x}.$$

Example.—Required the present value of an annuity of £1, £2, &c., on (55), the increase of which is to be arrested after 10 years.

Answer. $\frac{S_{55}-S_{65}}{D_{55}}=\frac{60976\cdot4989-17008\cdot1044}{587\cdot3514}=\frac{43968\cdot3945}{587\cdot3514}$

$=74\cdot8589=£74.\ 17s.\ 2d.$

We have now to deduce the expressions for the present values of the assurance benefits.

PROBLEM X.—To find the present value of an endowment assurance of £1 on (x); that is, of £1 to be received n years hence, provided (x) shall have died in the preceding year, viz., in the nth year from the present time.

Of the l_x individuals, whose present age is x years, represented by the mortality table to be now alive, l_{x+n-1} survive $n-1$ years, and l_{x+n} survive n years; consequently, $l_{x+n-1}-l_{x+n}$ is the number who die in their nth year; and it is also the number of pounds which will have to be paid, at the end of n years, to the representatives of those who thus die. The present value of this sum, therefore—that is, the sum which, put out at interest now, would in n years just amount to the first-named sum, is what must be advanced now to provide for this payment. This present value is $(l_{x+n-1}-l_{x+n})v^n$. And since all the l_x individuals now alive are equally interested, all of them contribute equally to this amount. The contribution of each will be, therefore—

$$\frac{(l_{x+n-1}-l_{x+n})v^n}{l_x},$$

and this is the required present value.

As in Problem I., multiply the numerator and the denominator of this expression by v^x, and it becomes

$$\frac{(l_{x+n-1}-l_{x+n})v^{x+n}}{l_x v^x}.$$

But the numerator of this expression is equal to C_{x+n-1}, and the denominator to D_x, by (1) and (3). Hence, the expression for the required present value, by the Commutation Table, is

$$\frac{C_{x+n-1}}{D_x}.$$

If $n=1$, that is, if the assurance is to be received a year hence, provided the death of x take place in the present year, the expression becomes $\frac{C_x}{D_x}$. If $n=2$, it is $\frac{C_{x+1}}{D_x}$, and so on.

Example.—Required the present value of an endowment assurance of £1 on (30), provided he die in his 40th year.

Here $x=30$, and $n=10$. Hence, the present value is $\frac{C_{39}}{D_{30}}$.

Since C is not exhibited, we may use for it either of the expressions (10) or (12). Taking the first, the formula becomes

$$\frac{M_{39}-M_{40}}{D_{30}}=\frac{540\cdot81079-522\cdot65022}{2257\cdot6521}=\frac{18\cdot16057}{2257\cdot6521}=\cdot008244=2d.$$

If the sum to be received be £100, its present value will be

$$\cdot008244\times100=\cdot8244=16s.\ 6d.$$

We have seen that when $n=1$, that is, when the sum assured is to be received a year hence, provided (x) be then dead, the formula for the present value becomes $\frac{C_x}{D_x}$. But, by (12), $C_x=vD_x-D_{x+1}$. Hence, by substitution, the formula in this case becomes

$$\frac{vD_x-D_{x+1}}{D_x}=v-\frac{D_{x+1}}{D_x};$$

and this expression affords a proof of the correctness of the formula in this problem. For, since v is the present value of £1, to be certainly received a year hence, and $\frac{D_{x+1}}{D_x}$ is (by Problem I.) the present value of £1 to be received at the same time provided (x) be then alive, the difference between these two is evidently the present value of £1 to be received a year hence if (x) be then dead.

Problem XI.—To find the present value of a life assurance of £1 on (x); that is, of £1 to be received at the end of the year in which (x) dies, whensoever that event may happen.

We saw, by the last problem, that $\frac{C_x}{D_x}$ is the sum which must be contributed now by each of the l_x individuals to provide for the payment of £1 to the representatives of each of those of their number who die in the first year; and that $\frac{C_{x+1}}{D_x}$ is the sum each must contribute to provide for a like payment to the representatives of each of those who die in the second year. And, in like manner, by making n successively equal to 3, 4, &c., to the end of life, in the formula deduced in last problem, we should obtain, for the amounts to be contributed now to provide for the deaths that take place in the 3rd, 4th, &c., years respectively,

$$\frac{C_{x+2}}{D_x},\ \frac{C_{x+3}}{D_x},\ \&c.$$

And the sum of all these expressions is evidently the total sum which each of the l_x individuals must now pay to entitle his representatives to £1 at the end of the year in which he dies.

This sum is

$$\frac{C_x + C_{x+1} + C_{x+2} + \&c.}{D_x},$$

which, by (2), is equal to $\frac{M_x}{D_x}$; and this is, therefore, the required present value.

Example.—Required the present value of a life assurance of £1 on (20).

$$\textit{Answer.} \quad \frac{M_{20}}{D_{20}} = \frac{1131\cdot76185}{3818\cdot9594} = \cdot296353 = 5s.\ 11d.$$

If the amount assured be £100, its present value will be

$$\cdot296353 \times 100 = 29\cdot6353 = £29.\ 12s.\ 8d.$$

PROBLEM XII.—To find the present value of a deferred assurance of £1 on (x); that is, of £1 to be received at the end of the year in which (x) dies, provided that event take place after n years.

The present value of this benefit, similar to that of the benefit in Problem III., is evidently equal to the sum of all the terms after the nth in the series of last problem, which expresses the present value of the same benefit for the whole life. This sum is, by (4), $\frac{M_{x+n}}{D_x}$; which is, therefore, the present value required.

Example.—Required the present value of an assurance of £1, deferred for 10 years, on (20).

$$\textit{Answer.} \quad \frac{M_{30}}{D_{20}} = \frac{742\cdot95673}{3818\cdot9594} = \cdot194544 = 3s.\ 11d.$$

If the assurance be £100, its present value will be

$$\cdot194544 \times 100 = 19\cdot4544 = £19.\ 9s.\ 1d.$$

PROBLEM XIII.—To find the present value of a temporary assurance of £1 for n years on (x); that is, of £1 to be received at the end of the year in which (x) dies, provided that event take place during the next n years.

The present value here is evidently the sum of the first n terms in the general series of Problem XI., which sum is, by (7),

$$\frac{M_x - M_{x+n}}{D_x};$$

and this is, therefore, the present value required.

Example.—Required the present value of a temporary assurance of £1 for the next 10 years on (20).

$$\textit{Answer.} \quad \frac{M_{20} - M_{30}}{D_{20}} = \frac{1131\cdot76185 - 742\cdot95673}{3818\cdot9594} = \frac{388\cdot80512}{3818\cdot9594} =$$
$$\cdot101809 = 2s.$$

If the sum assured be £100, its present value is

$$\cdot101809 \times 100 = 10\cdot1809 = £10.\ 3s.\ 8d.$$

If the present value just found be added to that found by the last problem, their sum, ·296353, is the present value of an assurance for the whole life, as found by Problem XI., which it evidently ought to be.

Problem XIV.—To find the present value of a deferred temporary assurance of £1 on (x); that is, of £1 to be received at the end of the year in which (x) dies, provided that event take place in the n years following the next k years.

The present value required in this problem is evidently the sum of n terms of the general series of Problem XI., beginning with the $(x+k)$th; which sum is, by (9), equal to

$$\frac{M_{x+k} - M_{x+k+n}}{D_x};$$

and this is, therefore, the present value required.

Example.—Required the present value of an assurance of £1 on (30), to be received at the end of the year of death, provided that event take place between the ages of 40 and 50.

Here $x=30$, $k=10$, and $n=10$.

$$\therefore \frac{M_{x+k} - M_{x+k+n}}{D_x} = \frac{M_{40} - M_{50}}{D_{30}} = \frac{742\cdot05673 - 522\cdot65027}{2257\cdot6521} = \frac{220\cdot30646}{2257\cdot6521}$$

$$= \cdot097582 = 1s.\ 11\tfrac{1}{2}d.$$

If the sum assured be £100, its present value will be

$$\cdot097582 \times 100 = 9\cdot7582 = £9.\ 15s.\ 2d.$$

Problem XV.—To find the present value of an increasing life assurance on (x), which is to be £1 if death take place in the first year, £2 if in the second, £3 if in the third, and so on—the assurance increasing each year that the payment is deferred, by the amount of the first year's assurance, till the end of life.

The present value here is found exactly in the same way as that of the corresponding annuity benefit in Problem VI.; for this assurance may be conceived to be made up of a series of assurances of £1, of which the first is to be entered upon immediately, and the others at intervals of one year, until the life shall have ceased to exist. The sum of the present values of all these assurances will evidently be the present value required. Now, the present value of the first is, by Problem XI., $\frac{M_x}{D_x}$; and of the second, third, fourth, &c., by Problem XII.,

$$\frac{M_{x+1}}{D_x}, \quad \frac{M_{x+2}}{D_x}, \quad \frac{M_{x+3}}{D_x}, \quad \text{\&c.};$$

and the sum of all these—that is, the present value required in the problem—is, by (2),

$$\frac{M_x+M_{x+1}+M_{x+2}+M_{x+3}+\text{\&c.}}{D_x}=\frac{R_x}{D_x}.$$

The same expression would be obtained by adding together the present values of the assurances for the separate years. Thus, the present value of £1 to be received at the end of one year, if death take place in that year, is, by Problem X., $\frac{C_x}{D_x}$; of £2 to be received at the end of two years, if death take place in the second year, $\frac{2C_{x+1}}{D_x}$; of £3 if death take place in the third year, $\frac{3C_{x+2}}{D_x}$, and so on. And the sum of these is, by (11), $\frac{R_x}{D_x}$, as before.

Example.—Required the present value of an increasing assurance of £1, £2, £3, &c., on (30).

Answer. $\frac{R_{30}}{D_{30}}=\frac{17127{\cdot}42459}{2257{\cdot}6521}=7{\cdot}5866=$ £17. 11*s.* 9*d.*

PROBLEM XVI.—To find the present value of an increasing assurance, deferred for n years, on (x), which is to pay £1, £2, £3, &c., at the end of the year in which (x) dies, according as that event shall take place in the $(n+1)$th, $(n+2)$th, $(n+3)$th, &c., year from the present time.

By reasoning precisely similar to that employed in Problem VII. for finding the present value of the corresponding annuity benefit, and which it seems, therefore, unnecessary to repeat, the expression for the present value of the benefit in the present problem is found to be $\frac{R_{x+n}}{D_x}$.

Example.—Required the present value of an increasing assurance of £1, £2, £3, &c., deferred 15 years, on (30).

Answer. $\frac{R_{x+n}}{D_x}=\frac{R_{45}}{D_{30}}=\frac{8326{\cdot}12414}{2257{\cdot}6521}=3{\cdot}6890=$ £3. 13*s.* 9*d.*

PROBLEM XVII.—To find the present value of a temporary increasing assurance of £1, £2, £3, &c., for n years, on (x).

The solution of this problem is strictly analogous to that of Problem VIII., and the required present value is found to be

$$\frac{R_x-R_{x+n}-nM_{x+n}}{D_x}.$$

Example. $x=30$, $n=5$.

Ans. $$\frac{R_{30}-R_{35}-5M_{35}}{D_{30}}=\frac{17127{\cdot}42459-(13672{\cdot}73323+3099{\cdot}47060)}{2257{\cdot}6521}$$
$$=\frac{355{\cdot}22076}{2257{\cdot}6521}={\cdot}15734=3s.\ 2d.$$

Problem XVIII.—To find the present value of an arrested increasing assurance of £1, £2, £3, on (x); that is, of an assurance which is to be £1 if death take place in the first year, £2 if in the second year, £3 if in the third, and so on to the nth year, when the assurance will be £n, at which amount it is to continue for the remainder of life.

The present value here, by a mode of proceeding similar to that in Problem IX., will be found to be

$$\frac{R_x-R_{x+n}}{D_x}.$$

Example. $x=30$, $n=5$.

Answer. $$\frac{R_x-R_{x+n}}{D_x}=\frac{R_{30}-R_{35}}{D_{30}}=\frac{17127{\cdot}42459-13672{\cdot}73323}{2257{\cdot}6521}$$
$$=\frac{3454{\cdot}69136}{2257{\cdot}6521}=1{\cdot}5302=£1.\ 10s.\ 7d.$$

The foregoing comprise all of what may be called the *simple* benefits, and some (we mean the increasing benefits) which perhaps do not, in strictness, belong to that category. Previous to making a few general and miscellaneous remarks upon them, we present the following synoptical table of the results we have obtained. We omit, for brevity and distinctness, the common divisor D_x.

The present Value of an	To be received in n years, is	For Life, is	For the first n years, is	After the first n years, is	For n years after k years, is	Uniform after n years, is
Endowment of £1 on (x)	D_{x+n}					
Endowment assurance of £1 on (x)	C_{x+n-1}					
Annuity of £1 on (x) ..	..	N_x	$\{N_x-N_{x+n}\}$	N_{x+n}	$\{N_{x+k}-N_{x+k+n}\}$	
Assurance of £1 on (x)	..	M_x	$\{M_x-M_{x+n}\}$	M_{x+n}	$\{M_{x+k}-M_{x+k+n}\}$	
Increasing annuity of of £1, £2, &c., on (x)	..	S_x	$\{S_x-S_{x+n}-nN_{x+n}\}$	S_{x+n}	$\{S_{x+k}-S_{x+k+n}-nN_{x+k+n}\}$	$\{S_x-S_{x+n}\}$
Increasing assurance of £1, £2, &c., on (x)	..	R_x	$\{R_x-R_{x+n}-nM_{x+n}\}$	R_{x+n}	$\{R_{x+k}-R_{x+k+n}-nM_{x+k+n}\}$	$\{R_x-R_{x+n}\}$

On looking at the expressions in the above table, the following analogies present themselves, attention to which will prevent mis-

takes in practice. We confine our remarks to the numerators, and it must be remembered that the denominator *in each case* is the D corresponding to the present age.

1. When a benefit for the whole life is to be entered upon immediately, the quantities in the numerator have for their signature the present age.

2. When the benefit is not to be entered upon immediately, the quantities in the numerator (which, as regards the columns made use of, are the same as in the expressions for the corresponding benefits to last the whole life), have for their signature the age at which the benefit is to be entered upon.

3. When the benefit is to be entered upon immediately, and to last for a term of years only—for example, n years—the expressions, as regards the uniform benefits, are derived from those for the same benefits to last the whole life, by writing $N_x - N_{x+n}$ and $M_x - M_{x+n}$ for N_x and M_x respectively; and as regards the increasing benefits, by writing $S_x - S_{x+n} - nN_{x+n}$ and $R_x - R_{x+n} - nM_{x+n}$ for S_x and R_x respectively. If we write $S_x - S_{x+n}$ and $R_x - R_{x+n}$ for S_x and R_x, we adapt the formulæ to the case in which the increase only is arrested at the end of n years.

4. The expression for an annuity benefit is changed into that for the corresponding assurance benefit, by writing C, M, and R, for D, N, and S respectively—the signatures remaining the same, except in the case of C, the signature of which must be diminished by unity.

Since the expressions we have deduced are fractions, the terms of which consist of numbers in the Commutation Table all of the same dimension, none of them rising above the first degree, it follows that, if all the numbers in the table be either multiplied or divided by any number whatsoever, the numerical values of the expressions will not be affected.

Hence the truth of the remark in the Note, page 93, is established.

And hence, also, it follows, that the position of the decimal point in the Commutation Tables is perfectly arbitrary. The position assigned to it in our table is that corresponding to a radix of the mortality table of 10,000, although the mortality actually made use of corresponds to a much higher radix, as stated in the previous part of this paper, p. 87.

The general problem solved in the preceding pages is, What is the *present value* of a benefit whose *amount* is £1? Suppose the problem reversed, and that it is required to find what *amount* of

benefit is equivalent to a *present value* of £1. The following proportion evidently subsists in all cases:—The present value of a benefit whose amount is £1 is to the present value of a like benefit of any other amount, as the amount of the first to the amount of the second. Hence, calling a the present value of the first benefit, and £1 that of the second, we have $a : 1 :: 1 : \frac{1}{a}$. That is, we should be furnished in each case with an expression which is the reciprocal of that which we have obtained for the present value of the same benefit when its amount is £1. Thus, $\frac{D_x}{N_x}$ will be the amount, or annual rent, of a life annuity to be entered upon immediately, and $\frac{D_x}{M_x}$ the sum to be received at the end of the year of death, in consideration of a present payment of £1, by an individual now aged x years. So, also, $\frac{D_x}{S_x}$ will be the first, and $\frac{2D_x}{S_x}$, $\frac{3D_x}{S_x}$, &c., the second, third, &c., annual payments of an increasing life annuity whose present value is £1.

Hence the table on page 177 furnishes us also with a solution to the general problem in this form; for, to apply it to any particular case, we have only to form a fraction having for its *numerator* D_x, and for its *denominator* the expression in the table corresponding to that case. But this belongs more properly to another branch of the subject.

Before concluding this portion of the present paper, we make a remark, which perhaps might with equal propriety, have been introduced elsewhere. It is as to the principle on which the preceding problems have been solved.

Each benefit consists of a payment or payments, to be made at a certain specified time or times, provided a state of things, also specified, shall then have place; such state of things being, where a single life is concerned, either the existence or the non-existence of that life, coupled, in the latter case, with the further condition, that its failure shall have taken place within a certain specified period. Problems I. and X. (using the former or the latter according as the first or the second of the states of things mentioned is that on which the receipt of the benefit is made to depend) enable us to find the present value of the payment for any or each year of existence; and the sum of the present values for each year that the benefit is to last is evidently the whole present value of the benefit. Thus, the present value of a life annuity of £1 is the

sum of a series, whose terms express respectively the present value of £1 to be received at the end of one year, of two years, of three years, &c., from the present age to the end of life. The present value of an annuity of £1 to last n years, is the sum of the first n terms of the same series; and that of an annuity of the same amount, to be entered upon n years hence, and to last during the remainder of life, is the sum of all the terms after the nth.

(*To be continued.*)

INSTITUTE OF ACTUARIES.

PROCEEDINGS OF THE INSTITUTE.

First Ordinary Meeting, Session 1861-62.—*Tuesday, 25th November*, 1861.

The President in the Chair.

The minutes of the annual general meeting were read and confirmed.

The Secretary announced various donations to the library.

Messrs. R. B. Markby and T. J. Searle, duly nominated at the last ordinary meeting, were unanimously elected Associates of the Institute.

Mr. W. Spens read a paper "On the mortality experience of the Scottish Amicable Life Assurance Society."

Thanks were voted to Mr. Spens, and the meeting adjourned to Monday, 30th December, 1861.

Second Ordinary Meeting, Session 1861-62.—*Monday, 30th December*, 1861.

The President in the Chair.

The minutes of the last ordinary meeting were read and confirmed.

The Secretary announced various donations to the library.

The undermentioned gentlemen, duly nominated at the last ordinary meeting, were unanimously elected members of the Institute, viz.:—

Official Associate—H. Harben, Esq.

Associates.

Mr. J. N. Smith.
„ J. B. Bourne.
Mr. S. C. Thomson.
„ T. Dence.

The President announced, that out of twelve candidates for the matriculation examination (1861), six had passed, in the following order of merit:—

Mr. J. N. Smith.
„ D. A. Bumsted.
„ F. J. C. Taylor.
Mr. T. J. Searle.
„ S. C. Thomson.
„ M. P. Christie.

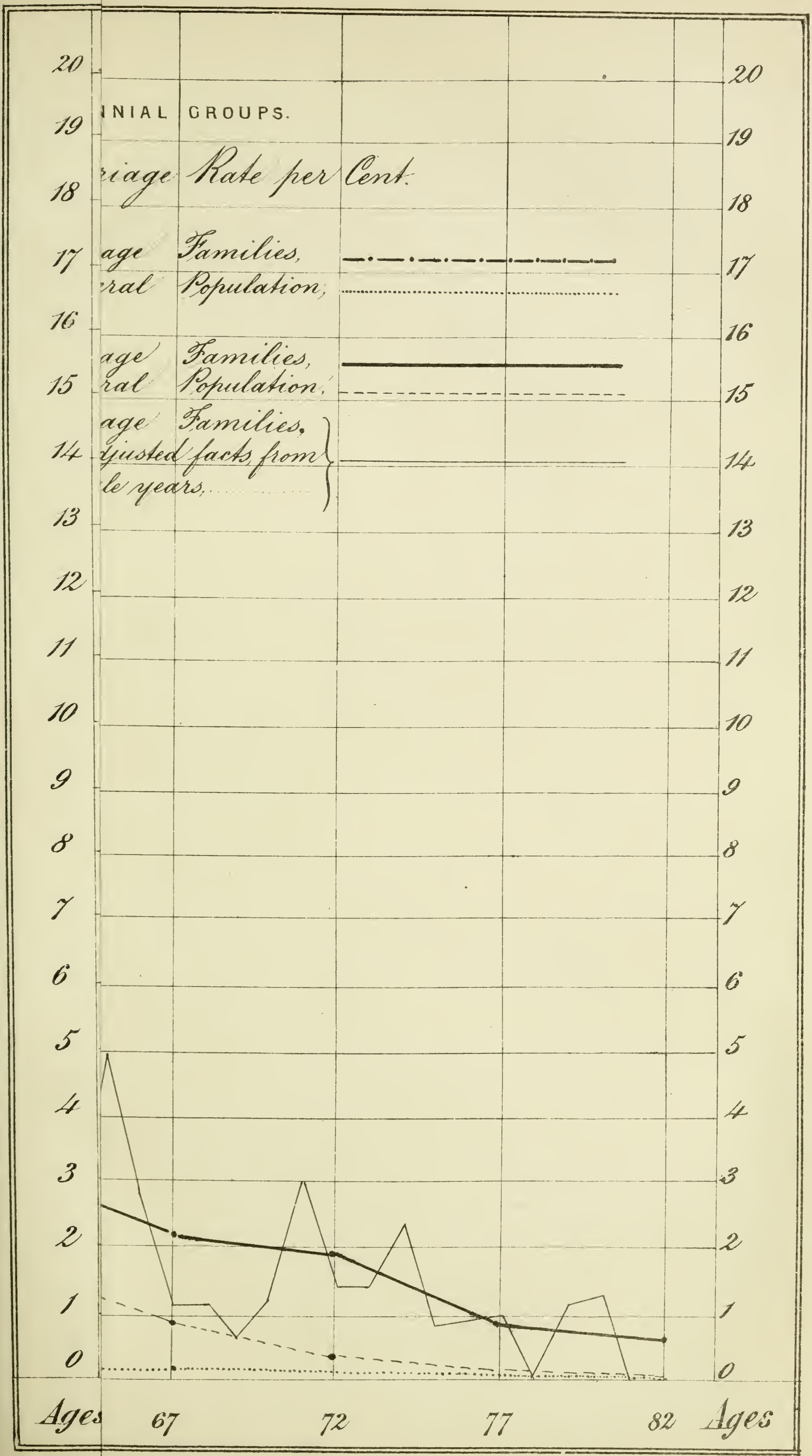
That seven candidates had presented themselves for the second year's examination, of whom three had passed in the following order:—

Mr. W. M. Makeham.
„ R. P. Hardy.
„ John Finlaison.

And that only one candidate had presented himself for the third year's examination, viz., Mr. Marcus Nathan Adler, M.A., who had also passed.

Mr. W. S. B. Woolhouse then read a paper "On Gompertz's law of mortality, and the dependence between it and Simpson's rule for calculating an annuity on three lives."

Thanks having been voted to Mr. Woolhouse, the meeting adjourned to Monday, 27th January, 1862.



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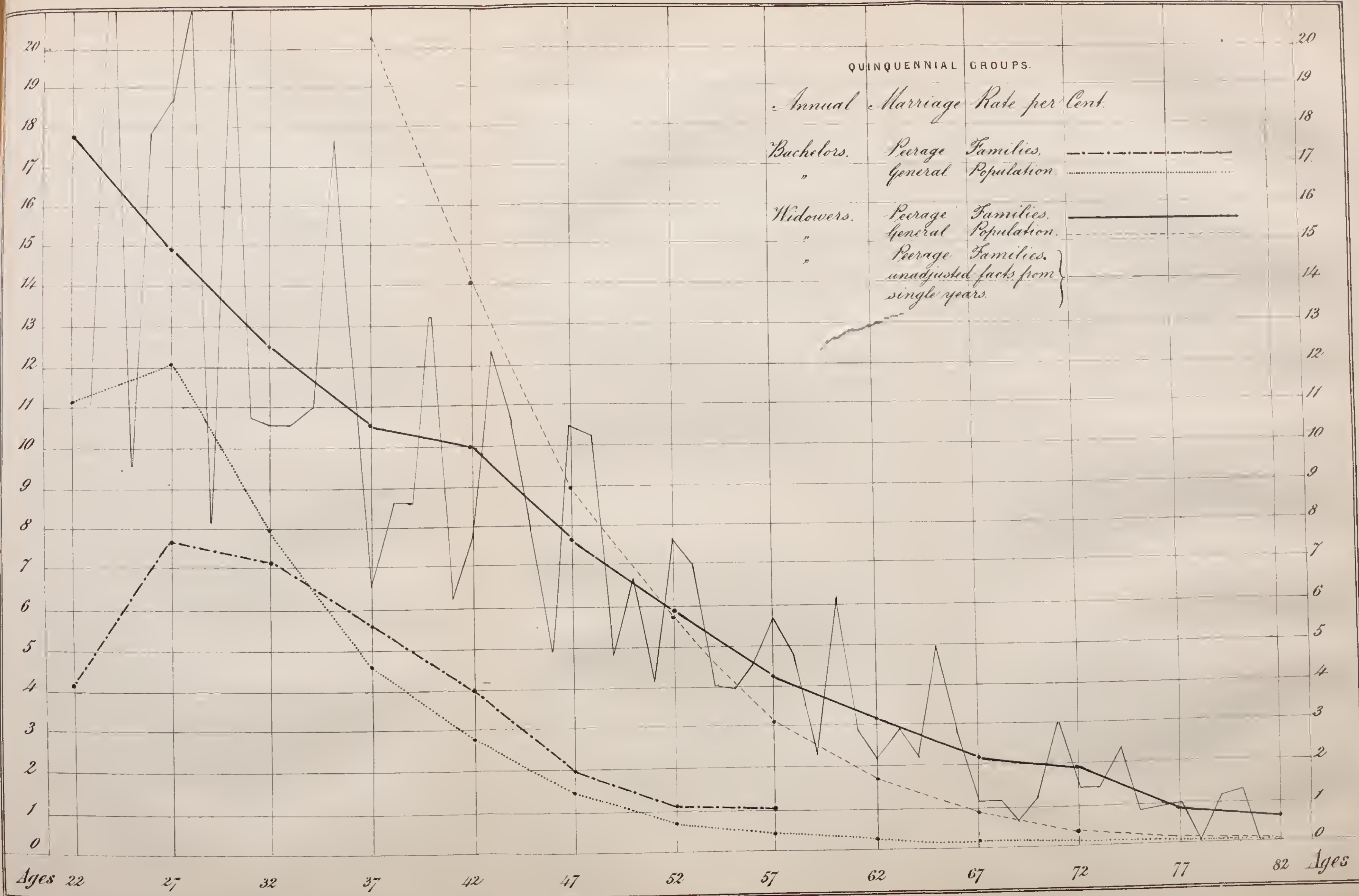
QUINQUENNIAL GROUPS.
Annual Marriage Rate per Cent.
Bachelors. Peerage Families.
" General Population.
Widowers. Peerage Families.
" General Population.
" Peerage Families. unadjusted facts from single years.
Ages
22
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THE

ASSURANCE MAGAZINE,

AND

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the Statistics of First and Subsequent Marriages among the Families of the Peerage, considered specially with reference to the Calculation of Premiums for Assurances against Issue. By ARCHIBALD DAY, *Esq., Actuary of the London and Provincial Law Assurance Society.*

[Read before the Institute, 24th March, 1862, and printed by order of the Council.]

IN the month of January, 1859, I had the honour to read before the Institute of Actuaries a paper upon the determination of the rates of premium for assuring against issue. Since that period the subject has lost none of its interest; on the contrary, its importance is yearly increasing, and it bids fair to form an important branch of assurance business. It is true that, out of the numerous inquiries made at Assurance Offices respecting the rates of premium for these risks, not 20 per cent. become the subject of policies absolutely granted, still it is desirable that there should be some sort of consistency in the premiums quoted. It would appear that considerable differences of opinion exist among actuaries upon the subject; and I feel, therefore, that I owe no apology to the Institute for departing from our usual routine subjects of rates of interest and mortality, and introducing once more the question of

issue policies and the statistics of marriage. On the former occasion we considered the case which is usually presented to us officially—namely, that in which a husband and wife are both living, but where the prospect of children by the existing marriage (hitherto unfruitful) is so very remote that it may be disregarded, and the assurance required is generally against the risk of issue to the husband by his present or any future wife. It is clear, from a consideration of the case, that, next to the question of survivorship between husband and wife, the most important feature is the probability of the husband's *re*marriage; and this is a matter concerning which we have at present but little information. But although the probabilities of marriage cannot be calculated with the same degree of precision as those of death and survivorship—the decease of every man being a certain event, whereas his marriage may be affected by various circumstances, and is, to a great extent, under his own control—yet it may be shown that a general law of marriage really does exist, and that, too, sufficiently definite for ordinary purposes. It is, however, unnecessary for me here to enter upon the argument, the subject having been exhaustively treated by abler pens than mine; and I need only refer to the labours of M. Quetelet, and to the more familiar papers of Mr. Samuel Brown, which are to be found in the early volumes of the *Journal* of the Institute.

The opinion has long been prevalent that a distinctive marriage rate would be found to prevail among different classes of the community, and that the aristocracy and landed gentry would exhibit a law differing materially from that of the labouring population. As the former are essentially the class whose reversionary interests frequently involve the necessity of issue policies, it seemed highly desirable that, as actuaries, we should endeavour to obtain some definite information as to the operation of that law—if, indeed, it should be found to exist at all.

The rate of mortality of this class, it will probably be considered, was satisfactorily shown in the results of the investigation made by my friend Mr. Bailey and myself upon the families of the peerage, which was presented to the Institute in the course of last Session; and it is from the confidence with which those observations inspired me in the accuracy and sufficiency of the data, that I have been induced now to make use of the same materials for determining the rate of marriage among the families of the aristocracy.

The facts on the present occasion have been extracted from the same book of the peerage as before—namely, that of Lodge—

assistance being also obtained from an older volume of Debrett. The date up to which the observations reach is 31st December, 1855, and they are almost all included within a period of 100 years preceding. It should be remarked that, in the observations upon the marriages of bachelors, no cases were extracted from the records of a family if the information as to the remainder was defective; that is to say, that if, for instance, in a particular family the necessary particulars as to the eldest son happened to be given, and they were deficient as to the other sons, that case was, perhaps, with an excess of caution, omitted from the list. On the other hand, as the facts relating to second and subsequent marriages were comparatively scanty, resource has occasionally been had to some of the collateral branches, but in instances only where the families were of such distinction that little risk of error was involved. The probabilities of the marriage of spinsters, and of the remarriage of widows, have not been omitted through any lack of interest in this—to the fairer portion of the community—all important question; but as this paper has been prepared with the avowed intention of applying the results to a particular case, dependent rather on the other sex, the consideration of their claims must be deferred to a more favourable opportunity.

Referring, then, to the male lives alone, the following are the results of the observations, which extended over 3,666 cases:—

	Bachelors.	Widowers.	Total.
Married	1,540	438	1,978
Died (unmarried). . . .	445	368	813
Existing on 31st December, 1855 (unmarried) . .	736	139	875
	2,721	945	3,666

It will be noticed that more than 50 per cent. of the whole number contracted marriage during the period under observation; and although the number of facts appears to be small, it will be remembered that tables of mortality have been founded upon a less number. The total number of deaths in the Carlisle Table was 1840 only, which is exceeded by the number of marriages before us; and the comparison is more effective when it is borne in mind that the chance of marriage, unlike that of death, extends over a portion only of the term of man's life. Few observations, upon a class at any rate, can boast of being more extensive; and I venture to submit, that the number of facts, when grouped, as will

be shown hereafter, in quinquennial periods of age, will be sufficient to give results upon which dependence may be placed. The total number of years during which the probability of marriage has been observed is 50,272; viz.—

For bachelors	42,383·5
„ widowers	7,888·5
	50,272·0

I should here explain that no distinction has been made between second and subsequent marriages—that is to say, a person who has become for the second time a widower has been regarded in the same light as the survivor of one wife only. I have not kept a record of the number of these cases, but, speaking from memory, I think that three came under observation who had each ventured upon a fourth marriage.

I now submit, in the following Tables I. and II., the facts observed under every year of age for bachelors and widowers, together with the annual marriage rate deduced therefrom. Owing, however, to the small numbers living at some ages, there are great inconsistencies in the results, and it has been thought desirable to show the facts also in groups of five years, from which the marriage rate deduced forms a table of great regularity. The rate for intermediate ages has been obtained from the first differences of the logarithms. The headings of the different columns will probably sufficiently explain their construction.

In Table I., perhaps, the first fact which arrests attention is, that, out of 2,721 males completing the age of 15, there are not 100 left at the age of 60 living in a state of celibacy; and that, out of that number of cases remaining, not one bachelor contracted a marriage after the age of 59. It is much to be regretted that the numbers under observation above that age are so small, since no marriage rate can be obtained from them. Of course, it does not follow that no marriages of bachelors take place in this class above 60, but I can find no facts from which to deduce any conclusions. According to the table of first marriages among the general population, in my previous paper, there were, for the quinquennial group between 55 and 60, but 3·5 marriages per 1,000, and from 60 to 65, 1·5 per 1,000 only. It might, however, be anticipated that the marriage rate in the peerage families at these ages would be greater, because, on a reference to the two previous quinquennial groups, it will be found to have been in

excess. This may, perhaps, appear more clearly hereafter, on reference to the diagram which accompanies this paper. The rate is apparently greatest at the age 27, being nearly 9 per 1,000; and the rate for the quinquennial group from 25 to 30 is greater than any other—perhaps, however, it would not have been anticipated that it would have been so nearly twice as great as between 20 and 25—while it very little exceeds that at the next higher ages, 30 to 35. It will be further noticed that, between the ages 35 and 40, the rate is greater than between 20 and 25. These results will, I think, warrant the conclusion that the marriages of the aristocracy do not, on the average, take place at such early ages as those of the general population. To this point I shall again have an opportunity of referring.

From Table No. II. (Widowers) it will be seen that the probabilities of widowers remarrying are throughout greater than those of the first marriages of bachelors, being nearly twice as great in the quinquennial groups 25–, 30–, and 35–, and continuing in an increasing ratio up to 50. From this point the comparison fails, in consequence of the deficiency in the numbers of first marriages. There is great regularity in the diminishing probabilities of marriage among the widowers, which, however, will best be seen by reference to the diagram, representing the annual marriage rate per cent., the average of each group of 5 years being taken to represent the rate at the middle age in the group. This is denoted by the dark even line, the irregular one showing the rate for every age from the unadjusted facts. The lower dark line exhibits the marriage rate for bachelors, also derived from the quinquennial groups; this, however, fails before the age of 60. The light dotted lines represent the marriage rate of bachelors and widowers respectively in the general population, and the comparison is shown in Table No. III., p. 190.

TABLE I.—*Peerage Families (Bachelors).*

Age (x).	Completed the Age (x).	Died between the Ages x and $x+1$.	Existing on 31st Dec., 1855, between the Ages x and $x+1$ (Bachelors).	Sum of two preceding Columns.	Number who might contract Marriage between Ages x and $x+1$.	Married between the Ages x and $x+1$.	Annual Marriage Rate.	Number who might contract Marriage. (Quinquennial Groups.)	Number Married. (Quinquennial Groups.)	Annual Marriage Rate. (Quinquennial Groups.)	Age (x).
15	2,721	9	31	40	2701·	1	·00037				
16	2,680	17	31	48	2656·						
17	2,632	15	29	44	2610·	2	·00077	13050·	25	·00192	17
18	2,586	17	23	40	2566·	4	·00156				
19	2,542	14	36	50	2517·	18	·00715				
20	2,474	20	46	66	2441·	25	·01024				
21	2,383	41	38	79	2343·5	94	·04011				
22	2,210	25	29	54	2183·	105	·04810	10865·	458	·04213	22
23	2,051	18	27	45	2028·5	114	·05620				
24	1,892	17	29	46	1869·	120	·06421				
25	1,726	21	24	45	1703·5	127	·07455				
26	1,554	23	30	53	1527·5	111	·07267				
27	1,390	11	15	26	1377·	122	·08860	6921·5	533	·07700	27
28	1,242	14	24	38	1223·	99	·08095				
29	1,105	14	15	29	1090·5	74	·06786				
30	1,002	18	19	37	983·5	73	·07422				
31	892	6	16	22	881·	55	·06243				
32	815	5	14	19	805·5	58	·07200	4048·5	289	·07137	32
33	738	7	15	22	727·	55	·07565				
34	661	5	14	19	651·5	48	·07368				
35	594	3	8	11	588·5	35	·05947				
36	548	10	10	20	538·	33	·06134				
37	495	6	10	16	487·	27	·05544	2466·5	135	·05473	37
38	452	5	10	15	444·5	22	·04949				
39	415	4	9	13	408·5	18	·04406				
40	384	6	7	13	377·5	23	·06093				
41	348	3	9	12	342·	11	·03216				
42	325	4	10	14	318·	15	·04717	1596·	63	·03947	42
43	296	3	12	15	288·5	6	·02080				
44	275	2	8	10	270·	8	·02963				
45	257	1	10	11	251·5	6	·02386				
46	240	2	5	7	236·5	6	·02537				
47	227	3	6	9	222·5	6	·02697	1112·5	22	·01977	47
48	212	2	10	12	206·	2	·00971				
49	198	1	3	4	196·	2	·01020				
50	192	2	8	10	187·	1	·00535				
51	181	1	5	6	178·	4	·02247				
52	171	..	6	6	168·	..	..	837·5	9	·01074	52

53	165	4	8	12	159·	3	·01887				
54	150	5	4	9	145·5	1	·00687				
55	140	2	6	8	136·	2	·01471				
56	130	5	9	14	123·	1	·00813				
57	115	2	3	5	112·5	1	·00889	573·5	6	·01046	57
58	109	4	5	9	104·5						
59	100	3	2	5	97·5	2	·02051				
60	93	1	2	3	91·5						
61	90	4	3	7	86·5						
62	83	4	2	6	80·	..	..	401·5			
63	77	3	2	5	74·5						
64	72	4	2	6	69						
65	66	1	4	5	63·5						
66	61	3	2	5	58·5						
67	56	5	3	8	52·	..	..	259·			
68	48	2	4	6	45·						
69	42	2	2	4	40·						
70	38	2	3	5	35·5						
71	33	..	4	4	31·						
72	29	3	2	5	26·5	..	..	137·5			
73	24	1	..	1	23·5						
74	23	2	2	4	21·						
75	19	2	2	4	17·						
76	15	..	1	1	14·5						
77	14	2	..	2	13·	..	..	66·5			
78	12	..	1	1	11·5						
79	11	1	..	1	10·5						
80	10	..	..	..	10·						
81	10	..	2	2	9·						
82	8	1	1	2	7·	..	..	36·			
83	6	1	..	1	5·5						
84	5	1	..	1	4·5						
85	4	..	1	1	3·5						
86	3	..	2	2	2·						
87	1	..	..	..	1·	..	..	8·5			
88	1	..	..	..	1·						
89	1	..	..	..	1·						
90	1	..	..	..	1·						
91	1	..	..	..	1·						
92	1	..	..	..	1·	..	..	3·5			
93	1	..	1	1	·5						
	42,974	445	736	1,181	42383·5	1,540		42383·5	1,540		

TABLE II.—*Peerage Families (Widowers).*

Age (x).	Completed the Age (x).	Died between the Age x and $x+1$.	Existing on 31st Dec., 1855, between the Ages x and $x+1$. (Widowers.)	Sum of two preceding Columns.	Number who might contract Marriage between Ages x and $x+1$.	Married between the Ages x and $x+1$.	Annual Marriage Rate.	Number who might contract Marriage. (Quinquennial Groups.)	Number Married. (Quinquennial Groups.)	Annual Marriage Rate. (Quinquennial Groups.)	Age (x).
20	1	..	..	..	1·						
21	3	..	..	..	3·						
22	3	..	..	..	3·	..	..	28·	5	·17858	22
23	9	..	..	..	9·	1	·11111			·17224	
24	12	..	..	..	12·	4	·33333			·16614	
25	21	..	..	..	21·	2	·09524			·16025	
26	28	..	..	..	28·	5	·17857			·15458	
27	38	1	..	1	37·5	7	·18667	194·5	29	·14910	27
28	48	..	1	1	47·5	10	·21053			·14390	
29	61	1	..	1	60·5	5	·08264			·13888	
30	72	2	..	2	71·	15	·21127			·13404	
31	75	..	1	1	74·5	8	·10738			·12936	
32	86	..	2	2	85·	9	·10588	408·5	51	·12485	32
33	87	1	2	3	85·5	9	·10526			·12051	
34	93	1	..	1	92·5	10	·10811			·11632	
35	104	1	2	3	102·5	18	·17561			·11227	
36	100	1	2	3	98·5	12	·12183			·10837	
37	110	2	3	5	107·5	7	·06512	554·5	58	·10460	37
38	118	1	1	2	117·	10	·08547			·10357	
39	131	3	1	4	129·	11	·08527			·10255	
40	137	1	..	1	136·5	18	·13187			·10153	
41	149	..	3	3	147·5	9	·06102			·10053	
42	157	1	..	1	156·5	12	·07668	763·5	76	·09954	42
43	164	2	2	4	162·	20	·12346			·09460	
44	163	3	1	4	161·	17	·10559			·08990	
45	166	5	5	10	161·	13	·08074			·08543	
46	167	4	..	4	165·	8	·04848			·08119	
47	175	2	4	6	172·	18	·10465	829·5	64	·07715	47
48	168	2	3	5	165·5	17	·10272			·07314	
49	168	2	2	4	166·	8	·04819			·06933	
50	171	8	3	11	165·5	11	·06646			·06573	
51	172	5	5	10	167·	7	·04192			·06231	
52	175	5	4	9	170·5	13	·07625	846·5	50	·05907	52
53	175	3	2	5	172·5	12	·06956			·05529	
54	173	1	3	4	171·	7	·04094			·05176	

55	179	4	3	7	175·5	7	·03989			·04845	
56	180	6	3	9	175·5	8	·04558			·04535	
57	176	1	1	2	175·	10	·05714	871·5	37	·04246	57
58	180	8	4	12	174·	8	·04598			·04029	
59	174	3	2	5	171·5	4	·02332			·03823	
60	180	3	3	6	177·	11	·06215			·03628	
61	174	7	..	7	170·5	5	·02933			·03443	
62	184	6	..	6	181·	4	·02210	887·5	29	·03268	62
63	183	10	..	10	178·	5	·02809			·03012	
64	184	3	3	6	181·	4	·02210			·02775	
65	193	13	7	20	183·	9	·04918			·02558	
66	179	8	5	13	172·5	5	·02899			·02357	
67	181	5	3	8	177·	2	·01130	874·5	19	·02173	67
68	184	18	1	19	174·5	2	·01146			·02112	
69	176	12	5	17	167·5	1	·00597			·02052	
70	168	9	2	11	162·5	2	·01231			·01995	
71	171	12	4	16	163·	5	·03067			·01939	
72	162	11	7	18	153·	2	·01307	743·	14	·01884	72
73	146	12	4	16	138·	2	·01449			·01597	
74	135	12	5	17	126·5	3	·02372			·01354	
75	124	17	4	21	113·5	1	·00881			·01148	
76	115	12	4	16	107·	1	·00935			·00973	
77	105	12	3	15	97·5	1	·01026	485·	4	·00825	77
78	96	12	4	16	88·					·00799	
79	84	8	2	10	79·	1	·01266			·00773	
80	78	11	1	12	72·	1	·01389			·00749	
81	68	11	1	12	62·					·00725	
82	58	4	1	5	55·5	..	..	285·	2	·00702	82
83	57	9	..	9	52·5	1	·01905				
84	48	10	..	10	43·						
85	43	11	1	12	37·						
86	32	8	1	9	27·5						
87	24	7	1	8	20·	..	..	105·	0	..	87
88	16	5	2	7	12·5						
89	10	4	..	4	8·						
90	6	1	..	1	5·5						
91	5	1	..	1	4·5						
92	4	4	..	4	2·	..	..	12·	0	..	92
	8,142	368	139	507	7888·5	438	..	7888·5	438		

TABLE No. III.

Age.	Probability of Marrying in a Year.			
	Bachelors.		Widowers.	
	General Population.	Peerage Families.	General Population.	Peerage Families.
15–	·00464	·00192		
20–	·11209	·04213	·30766	·17858
25–	·12209	·07700	·35791	·14910
30–	·07851	·07137	·28627	·12485
35–	·04558	·05473	·20313	·10460
40–	·02798	·03947	·14075	·09954
45–	·01448	·01977	·08858	·07715
50–	·00705	·01074	·05711	·05907
55–	·00349	·01046	·03201	·04246
60–	·00152	..	·01745	·03268
65–	·00146	..	·00862	·02173
70–	·00031	..	·00316	·01884
75–	·00059	..	·00100	·00825
80–	·00000	..	·00067	·00702

The column of "General Population" I take from my previous paper; it was derived from the number of marriages in the year 1851 according to the Registrar-General's returns, compared with the population according to the census of that year. The number of marriages in which the ages of both husband and wife were known was 56,347, and it was assumed that, in the cases of those whose ages were not known, the numbers in each quinquennial group would bear the same proportion to the total number as in the instances where the ages were given, and they were accordingly raised in that proportion. From this it appears that the first marriages of the aristocracy are at a much later age than those of the general population—the probability of marriage between 15 and 25 being less than one-half, and not much exceeding one-half between 25 and 30. At this period they become about equal, and for the remainder of life, so far as the table extends, the marriage rate of the peerage families is in excess.

The same peculiarity attaches to the second and subsequent marriages, the probability of remarriage appearing, under 40, to be only half, about 50 to be equal to, and above that age greater, in an extraordinary degree, than in the general population. The very great differences at the older ages in this class of second marriages would seem to require some observation, the distinction in rank seeming insufficient to account for it, although the attraction of a title may go some way to explain the readiness with which the aged nobility occasionally obtain second partners; and it appears to me possible that the error, if any there be, may be in the construction of the former table; for as it seems probable that

the proportion of persons in advanced years who would decline to acknowledge their ages at the time of marriage would be greater than of younger men, the distribution of the marriages of those whose ages were not given should not have been made in the same proportions.

That the marriages of the males of the peerage families take place later in life may again be shown by a comparison (see Table IV.) of the percentages of marriages at different periods of life with other tables previously published; and I allude especially to those in Mr. Brown's paper, *Assurance Magazine*, vol. vii.

TABLE NO. IV.—*Proportion per Cent. of Marriages.*

Ages.	PEERAGE FAMILIES.	ENGLAND (S. BROWN).		BELGIUM.	MASSACHUSETTS.	POORER CLASSES (ST. GEORGE'S-IN-THE-EAST).	PEERAGE FAMILIES.	1,027 PEERS (SADLER).
	First and subsequent Marriages.	1846–7–8.	1851–2–3.	1841–5.	$6\frac{3}{4}$ Years to January, 1857.	Statistical Society's Journal.	First Marriages only.	First Marriages only.
Under 30	53·08	76·77	75·62	52·75	75·01	85·00	65·97	62·81
30 to 45	33·98	18·31	19·22	39·93	19·54	14·06	31·63	27·75
45 to 60	9·50	4·03	4·25	6·05	4·12	·94	2·40	7·88
60 and upwards	3·44	·89	·91	1·27	1·33	0·00	0·00	1·56
	100·00	100·00	100·00	100·00	100·00	100·00	100·00	100·00

These Tables, with the exception of Mr. Sadler's (Col. 8), refer to both first and subsequent marriages; but as there are, perhaps, in the present observations, a larger proportion of widowers than in the others, I have added also the (Col. 7) the percentage of first marriages alone. Col. 1, then, shows the percentages of marriages according to the families of the peerage; Cols. 2 and 3 represent England as deduced by Mr. Brown; Col. 4 is from M. Quetelet's observations in Belgium; Col. 5 from the Registration Tables of Massachusetts; Col. 6 is derived from a Report of a Committee of the Statistical Society upon the poorer inhabitants of St. George's-in-the-East in 1845; and Col. 8 was compiled by Mr. Sadler, and is to be found in the second volume of his work on the *Law of Population*. It will be observed that the greatest similarity exists between the marriages in Belgium and the peerage families, for which at present I am unable to offer any satisfactory explanation; and that the greatest differences are between the two extremes in the social scale, the peerage and the poor of St. George's-in-the-East. The number of facts from which the latter table was

formed was 1,488. Mr. Sadler's facts are decidedly confirmatory of my own observations, but he has succeeded in obtaining greater numbers at the advanced ages. In his book, above referred to, he makes constant reference to his synoptical register of the peerage in support of his arguments against Mr. Malthus, but I am not aware that he has published it separately. Upon this part of my subject I need hardly, perhaps, dwell farther, and I pass on to some considerations as to the ages and condition of the ladies who became the second wives of the noble widowers.

The number of ladies whose ages at marriage were ascertained is 111 only, or 25 per cent. upon the whole number, a proportion so small that the following table cannot, I fear, be considered of much practical value.

Table No. V.—*Second Marriages of Widowers, with Ages of Wives.*

Ages of Husbands.	Ages of Wives.								
	15—	20—	25—	30—	35—	40—	45—	50—	55—
20—	1								
25—	1	5							
30—	2	9	1	1					
35—	4	7	5	2	2				
40—	2	8	7	4	2	1			
45—	3	3	7	1	3	1			
50—	1	1	4	..	2				
55—	..	2	3	3	1	1			
60—	..	1	..	2	..	..	1	..	1
65—	1	..	1	1	..	..	1	1	
70—	..	..	..	..	..	1			
75—									
80—									
	15	36	28	14	10	4	2	1	1

The ages of the husbands and second wives, it will be seen, are given in quinquennial groups, as before; and I would draw attention to two conclusions which may be deduced therefrom, which have some bearing on the question of assurances against issue, viz.—

1. That in no instance did the age of the second wife exceed that of her husband.
2. That in two instances alone did widowers contract marriage with ladies who had passed the ordinary period of child-bearing, assuming that to be the age of 50 years.

Of the 438 remarriages of widowers, it appears that 355, or 81 per cent., were contracted with spinsters; 81, or 18½ per cent.,

with widows; and of the remaining 2, one had been divorced from a previous husband, and the condition of the other, who was a foreign lady, could not be traced.

The duration of time between the death of a first wife and a subsequent marriage is a question of some interest, and I have accordingly constructed a table (No. VI.) of the average difference in years between the two events.

TABLE No. VI., *showing the Average Number of Years between Death of Wife and Remarriage of Widower.*

Age of Widowers.	Number of Widowers who remarried.	Number of Years between Death of Wife and Remarriage.	Average Number of Years.
20—	12	45·	3·75
25—	61	272·5	4·47
30—	62	369·	5·95
35—	72	359·	4·99
40—	66	290·	4·39
45—	49	196·5	4·01
50—	42	137·	3·26
55—	33	108·5	3·29
60—	21	76·5	3·64
65—	9	30·5	3·39
70—	7	11·5	1·64
75—	3	13·5	4·50
80—	1	·5	·50
	438	1910·	4·36

In numerous instances the period of mourning was very limited, while in two or three particular cases it might have been said, with Hamlet—

> "The funeral baked meats
> Did coldly furnish forth the marriage tables."

It will be seen that the average period between the two events is rather less than $4\frac{1}{2}$ years, but there is but little regularity in the results at different ages. It is a noticeable fact, that, out of 438 second and subsequent marriages, no less than 51, or 11·6 per cent., took place within a year from the termination by death of the first; and 113, more than 29 per cent. of the remainder, were contracted in the second year. In many instances it is to be presumed that virtual divorces had previously taken place, as we could otherwise hardly conceive the possibility of a second marriage contracted within *one week*. The extreme case on the other side is that of a nobleman who remained a widower for a period of $37\frac{1}{2}$ years and remarried at the advanced age of 68.

I had cherished a hope that in this investigation I might have been able to obtain some trustworthy information as to the fruitfulness of second marriages, as it has an important bearing upon the class of assurances against issue; but it would have been worse than useless to have attempted it with the few facts under observation. The ages of the parents form an important item in the investigation, and, unfortunately, there are but 111 cases in which the ages of both husband and wife are known; then, before the particulars could be complete for consideration, the duration of the marriage must have terminated by the death of one parent; and a third important consideration presents itself in the question, whether there had been children by the previous marriage. In the face of these difficulties I have, for the present, felt obliged to relinquish the attempt. Mr. Sadler, speaking generally, states that the average number of children from second and subsequent marriages is 2·75, and he observes that the children of these marriages are in a greater proportion females. How far his observations may be entitled to confidence, in the absence of the data upon which he grounded his conclusions, I am not prepared to give an opinion. I am informed that some new facts have very recently been published by the Registrar-General of Scotland, which may have an important bearing on the subject, especially as he infers that the proportion of unfruitful marriages increases with a higher rank in life; but this part of the question may well stand over for further consideration.

I may now apply some of the foregoing results to the practical question of issue premiums, and I have accordingly constructed Table VII. upon the same principle as on the former occasion, the assurance being for the sum of £1 payable at the end of the year in which a husband whose wife is now living may, after her decease, contract a second marriage. I need not enter upon the details of its construction, as they are already recorded in the *Assurance Magazine*, vol. viii.; suffice it to say, that the value of the benefit in the nth year consists of the product of the respective probabilities of the death of the wife before the end of that year, and of the survivorship of the husband, multiplied by the annual marriage rate at the advanced age—the whole discounted by the present value of £1 at the end of n years. A summation of these values for every age produces the required single premium. The elements of the table, with the exception of the marriage rate, are the same as before—namely, English Life Table No. 1—and a rate of interest £3 per cent.

Table No. VII.—*Present Value of £1 at the end of the Year in which a Husband shall contract a Second Marriage.*

Age of present Wife.	Age of Husband. 50.		55.		60.		65.		70.		75.	
	Peerage Families.	Former Observations.	Peerage Families.	Former Observations.	Peerage Families.	Former Observations.	Peerage Families.	Former Observations.	Peerage Families.	Former Observations.	Peerage Families.	Former Observations.
45	·0637	·0276	·0375	·0123	·0208							
50	·0782	·0323	·0458	·0145	·0251	·0056	·0128	·0018				
55	·1042	·0446	·0618	·0196	·0342	·0076	·0172	·0024	·0077	·0006		
60	..	..	·0842	·0276	·0473	·0108	·0244	·0035	·0111	·0009	·0039	·0002
65	..	..	..	..	·0643	·0150	·0338	·0049	·0156	·0013	·0055	·0003
70	..	..	..	..	..	..	·0456	·0068	·0215	·0018	·0077	·0004
75	..	..	..	..	..	..	..	..	·0289	·0025	·0106	·0006
80	..	..	..	..	..	..	..	..	..	..	·0144	·0008

From what has been said, it will at once have been anticipated that a very considerable increase in the rates formerly quoted has taken place, more especially for cases in which the lives are of advanced age; and since at those ages the annual marriage rate is 3, 4, and even 8 times greater, and as it enters into calculation for every year of life from 82 downwards to the present ages of the proposed assured, the extent of increase, though at first sight startling, will be seen to be entirely justified. An examination of the above table will show that more risk attaches to these transactions than, perhaps, we have lately been disposed to allow, the business having hitherto consisted chiefly of receiving premiums unaccompanied by the payment of corresponding claims; it is, however, submitted, that the limit of the risk has now been shown—the probabilities of remarriage can hardly be expected to exceed those of the peerage families, and there is still in favour of the Assurance Companies the possibility of no issue by the second marriage; in some instances also the risk is confined to the birth of male issue only, and in many there is the further chance of that issue dying before the parent, or before attaining the age of 21 years.

I have calculated, as a final table, the single premium for an assurance payable in the event of the subsequent marriage of a widower, the predecease of the wife having already taken place. This, it is considered, calls for no special remark, the case seldom occurring in practice without considerable modifications.

TABLE No. VIII.—*Single Premium for the Assurance of* £1 *payable on the Second Marriage of a Widower.* (*English Life Table; Interest,* £3 *per Cent.*)

Age.	Single Premium.	Age.	Single Premium.
50	·5463	65	·1471
55	·3666	70	·0879
60	·2397	75	·0384

In closing this paper I cannot but feel that its defects are numerous—that more minute details might have increased its value—and that many deductions as to the social condition of an important class might have been drawn by a more philosophical mind; but, while apologising for my shortcomings, I trust that my subject may have been one of interest, and that some little practical value may attach to my second attempt to determine rates of premium for a special class of risks.

Observations, by WILLIAM SPENS, *on Mortality Experience, supplementary to those contained in No. XLVI. of the Assurance Magazine.*

I HOPE, from time to time, to be able to give some further tables bearing on the subject, and to state how far I think subsequent experience and new information confirm or modify the interim conclusions which I ventured to submit in the present state of our knowledge; and I shall be very glad that my recent observations may receive attention, and correction where required, from critics abler than their author.

I have not observed any abstract exhibiting distinctly the general divisions of the 83,905 policies embraced by the Experience of Seventeen Offices; and, indeed, such abstract can only be made out by inferring some of the details from others. I now submit a table of this nature.

TABLE NO. I.—*Abstract of Data of Experience of Seventeen Offices, embracing* 83,905 *Policies.*

	(1) Number of Policies.	(2) Total Numbers exposed to a Year's Risk at each Age.	(3) Total Deaths.	(4) Average Duration of Policies.	(5) Percentage of Mortality on Col. 2.	(6) Discontinued.	(7) Existing.
Male Lives (Town), (B1)	16,097	89,601	1,190	5·6	1·3	4,457	10,450
Female Lives (Town), (B4) ..	1,448	6,848	134	4·7	1·8	447	867
D3, deduced from	17,545	96,449	1,324	5·5	1·4	4,904	11,317
London Equitable	21,398	262,210	5,144	12·3	2·0	9,324	6,930
London Amicable	4,618	55,087	1,792	11·9	3·2	535	2,291
Other Town Experience necessarily inferred to produce next line	5,141	59,914	914	11·7	1·5		
Combined Town Experience, Table E	48,702	473,660	9,174	9·7	1·9		
Country, D4 (B2 and B5) .	13,335	72,985	1,158	5·3	1·6	3,284	9,393
Table G, deduced from	62,537	546,645	10,332	8·7	1·9		
Irish, D5 (B3 and B6)	9,236	51,621	1,446	5·6	2·8	3,038	4,752
Other Experience necessarily inferred to produce next line..................	12,132	113,897	2,003	8·6	1·8		
Total Experience	83,905	712,163	13,781	8·5	1·9		
Total Experience inferred..	17,273	173,811	2,917	10·1	1·7	4,162	10,194
						25,247	44,877

The marked excess of mortality exhibited by the experience among the females and Irish lives is very distinctly shown in this abstract, and I think if the experience had shown nothing else than this, it would have been of the greatest practical importance to Offices. The attention thus directed to the necessity of more caution in regard to these risks has, I am confident, saved many tens of thousands of pounds to Assurance Offices.

Table D3 of the experience of Offices shows the number exposed to risk of male lives (town) and female lives (town) combined. I have separated "male lives (town)," and shown them, in Table No. II., with the total expected deaths by the English Life Table No. 2,* which have been calculated in detail. I have also divided the lives at risk, and the expected deaths during each year of the assurances, as per Tables III. and IV. These do not agree with the approximate statements I have made in the previous Number of the *Magazine* in reference to deaths among "male lives (town)" during and after the first six years of the assurances, owing to a misunderstanding of the position of the "existing" in that portion of the experience tables on which the calculations were founded; and I am glad to have this opportunity of making the correction, though it does not make any alteration on my opinion of the general conclusions.

Table No. II., *showing, in the case of "Male Lives (Town)" of the Experience of Offices, the Total Numbers exposed to the Risk of Mortality, the Actual Deaths, and the Expected Deaths by the English Life Table No. 2.*

Age.	Number Exposed to the Risk of Mortality.	Deaths.	
		Actual.	Expected by English Life Table No. 2.
Under 20	1,027	5	6·603
20 and under 25	2,373	20	20·072
25 „ 30 ...	7,463·5	51	68·914
30 „ 35	13,666·5	102	139·628
35 „ 40	16,647·5	151	191·565
40 „ 45	15,536·5	176	208·925
45 „ 50 .. .	12,680	167	202·721
50 „ 55	9,200	166	177·689
55 „ 60	5,825·5	145	146·434
60 „ 65	3,168·5	95	117·697
65 „ 70	1,443	66	75·221
70 „ 75	439	32	31·647
75 „ 80	106·5	11	11·447
80 „ 85 ...	24·5	3	3·849
	89,601	1,190	1,402·412

* It may be noted, that, throughout, it is the "male" table which is referred to.

TABLE No. III., *showing, in the Case of the "Male Lives (Town)" of the Experience of Offices, the Numbers exposed to the Risk of Mortality during each Calendar Year of the Assurances, and the Actual Deaths.*

Ages at Entry.	1st Year. (half.)		2nd.		3rd.		4th.		5th.		6th.		7th.		8th.		9th.		10th.		11th.	
	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.
Under 20 . . .	202	1	353½	2	269½	2	207	2	166	1	134½	2	112½	..	98½	..	84½	1	66	..	51	1
20 and under 25	602½	1	999	12	774	9	616	4	516½	1	422½	3	341	4	279½	1	231	..	197½	2	159½	1
25 „ 30	1,298	6	2,259	11	1,886	17	1,600½	17	1,372½	6	1,223	10	1,048	8	891	12	748	5	634	11	523½	5
30 „ 35	1,535	8	2,717½	19	2,317	22	1,960	14	1,698	15	1,467½	19	1,279	13	1,091	20	925½	12	785	11	658½	8
35 „ 40	1,435½	11	2,496½	24	2,100	23	1,758½	9	1,520½	15	1,350	11	1,186	21	1,008	14	838½	9	711½	7	593½	5
40 „ 45	1,132½	4	2,016	22	1,695	15	1,466	22	1,253½	10	1,084	22	938	12	806	18	681	8	581½	19	476	7
45 „ 50	789	10	1,383½	11	1,170	14	991	12	844½	16	734½	16	624½	17	526	10	442½	7	378½	8	329	8
50 „ 55	500½	3	865½	13	737	27	614	13	515	9	445	17	384	11	316½	7	256½	6	218	5	186	2
55 „ 60	304	5	520	13	443½	7	374½	16	312	5	270½	6	225	9	184½	3	151½	15	122½	6	102	6
60 „ 65	134½	3	227½	11	174½	4	144½	4	124	7	101	2	87½	2	76½	7	58½	3	47½	3	41	4
65 „ 70	44½	..	72½	4	59	4	45	4	33	1	30	3	20½	1	17	2	13	1	12	1	10	1
70 „ 75	12	1	18	..	14	1	10	..	10	1	8	..	6	2	4	..	3	..	3	..	2	1
75 and upwards	3½	1	3	..	2	..	2	..	2	..	1½	..	1	..	1	..	1	1	..	..	..	..
	7,993½	54	13,931½	142	11,641½	145	9,789	117	8,367½	87	7,272	111	6,253	100	5,299½	94	4,434½	68	3,757	73	3,132	49

Table No. III. (continued).

Ages at Entry.	12th.		13th.		14th.		15th.		16th.		17th.		18th.	
	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.
Under 20 ...	38	..	29½	..	17	..	8	..	5	..	5	..	2	..
20 and under 25	129½	1	96½	..	45	1	20	1	9	..	5	1	2	..
25 " 30	440	3	350½	2	211	..	107	2	59	1	37	1	22	..
30 " 35	556	8	450½	11	296	2	150½	1	85	2	47	1	34	..
35 " 40	507	4	427	9	279	3	146	..	87	3	39	..	28	..
40 " 45	390½	6	318	6	222½	3	121½	3	71	..	39½	5	19	1
45 " 50	266	6	221	9	133	3	66	3	42	1	20	1	16	2
50 " 55	157½	4	122	6	78	4	54	3	27½	2	15	1	5	..
55 " 60	69½	3	44½	3	25	..	18	2	12	1	7	..	4	2
60 " 65	28	1	20	3	12	..	7	1	4	..	2	1	..	..
65 " 70	7	..	6	..	4	1	2	..	1	..	1	..	1	..
70 " 75	1	..	1	..	1	..	..	..	..	..	..	..	..	..
75 and upwards	..	..	..	..	..	..	..	..	..	..	..	..	..	..
	2,590	36	2,086½	49	1,323½	17	700	16	402½	10	217½	11	133	5

Ages at Entry.	19th.		20th.		21st.		22nd.		23rd.		24th.		Totals.	
	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.	At Risk.	Died.
Under 20 ...	..	..	..	..	..	..	..	..	..	..	..	..	1,849·5	12
20 and under 25	1	..	1	..	1	..	1	..	..	..	..	..	5,450	42
25 " 30	16	..	8½	..	6	..	3	..	1	..	..	..	14,744·5	117
30 " 35	20	1	12½	..	9	..	7	..	4	..	4	..	18,109·5	187
35 " 40	20	..	19	1	13	1	11	..	10	..	7	..	16,591·5	170
40 " 45	13	..	9	..	8	..	5	..	4	1	3	..	13,353·5	184
45 " 50	10	..	10	..	8	..	7	..	6	..	4	..	9,022	154
50 " 55	4	1	3	..	2	..	1	..	1	..	1	..	5,509	134
55 " 60	2	..	1	1	..	..	..	..	..	..	..	..	3,193	103
60 " 65	..	..	..	..	..	..	..	..	..	..	..	..	1,290	56
65 " 70	..	..	..	..	..	..	..	..	..	..	..	..	378·5	23
70 " 75	..	..	..	..	..	..	..	..	..	..	..	..	93	6
75 and upwards	..	..	..	..	..	..	..	..	..	..	..	..	17	2
	86	2	64	2	47	1	35	..	26	1	19	..	89,601	1,190

TABLE No. IV., *showing, in the Case of the Lives at Risk in Table III., the Expected Deaths by the English Life Table No. 2.*

Ages at Entry.	1st Year. (half.)	2nd.	3rd.	4th.	5th.	6th.	7th.	8th.	9th.	10th.	11th.	12th.	13th.
Under 20	1·343	2·421	1·851	1·450	1·196	1·026	·881	·790	·702	·568	·449	·341	·270
20 and under 25	5·096	8·606	6·803	5·516	4·714	3·932	3·237	2·714	2·292	2·004	1·653	1·374	1·054
25 „ 30	11·955	21·224	18·091	15·682	13·750	12·529	10·991	9·572	8·201	7·150	6·082	5·264	4·329
30 „ 35	15·649	28·351	24·775	21·381	19·009	16·887	15·149	13·298	11·687	10·238	8·869	7·736	6·482
35 „ 40	16·491	29·511	25·555	22·152	19·762	18·128	16·478	14·491	12·480	10·985	9·522	8·452	7·391
40 „ 45	15·237	28·071	24·440	21·884	19·402	17·404	15·634	13·958	12·249	10·883	9·251	7·889	6·715
45 „ 50	12·633	23·001	20·201	17·782	15·775	14·257	12·619	11·097	9·842	8·970	8·343	7·284	6·556
50 „ 55	9·645	17·341	15·379	13·496	12·046	11·137	10·314	9·201	8·046	7·337	6·712	6·107	5·083
55 „ 60	7·565	13·972	12·926	11·762	10·569	9·815	8·759	7·783	6·900	6·038	5·412	3·970	2·684
60 „ 65	4·924	8·984	7·367	6·508	5·970	5·215	4·867	4·562	3·702	3·243	3·022	2·224	1·723
65 „ 70	2·247	3·902	3·398	2·797	2·191	2·146	1·574	1·392	1·141	1·133	1·020	·752	·689
70 „ 75	·919	1·499	1·242	·957	1·032	·892	·719	·523	·440	·471	·322	·149	·160
75 and upwards	·399	·359	·232	·249	·268	·218	·160	·171	·183	..	..	..	..
	104·103	187·242	162·260	141·616	125·684	113·586	101·382	89·552	77·865	69·020	60·657	51·542	43·136

Ages at Entry.	14th.	15th.	16th.	17th.	18th.	19th.	20th.	21st.	22nd.	23rd.	24th.	TOTALS.
Under 20	·160	·076	·047	·048	·021	..	..	..	..	..	..	13·640
20 and under 25	·497	·228	·107	·060	·025	·013	·013	·013	·014	..	..	49·965
25 „ 30	2·699	1·419	·809	·523	·323	·245	·131	·095	·066	·019	..	151·149
30 „ 35	4·410	2·324	1·356	·774	·585	·356	·236	·178	·144	·086	·091	210·051
35 „ 40	5·037	2·736	1·691	·789	·593	·445	·452	·327	·300	·292	·227	224·287
40 „ 45	4·979	2·893	1·840	1·107	·582	·429	·320	·303	·203	·177	·145	215·995
45 „ 50	4·270	2·313	1·597	·825	·710	·485	·524	·460	·429	·391	·276	180·640
50 „ 55	3·506	2·575	1·462	·852	·296	·248	·205	·138	·072	·077	·083	141·358
55 „ 60	1·627	1·246	·907	·553	·353	·202	·112	..	..	..	..	113·155
60 „ 65	1·117	·703	·433	·243	..	..	..	..	..	..	..	64·807
65 „ 70	·500	·268	·149	·160	·171	..	..	..	..	..	..	25·630
70 „ 75	·171	..	..	..	..	..	..	..	..	..	..	9·496
75 and upwards	..	..	..	..	..	..	..	..	..	..	..	2·239
	28·973	16·781	10·398	5·934	3·659	2·423	1·993	1·514	1·228	1·042	·822	1402·412

I beg now to submit, to be substituted for similar statement in the middle division of Table VII. of my paper in No. XLVI., the following—

Table No. V., *showing, in reference to "Town Males," the Deaths Actual and Expected by English Life No. 2, during first six Years and thereafter; also the Percentage of the Deaths, Actual and Expected, on the Lives exposed to Risk, and of the Actual on the Expected.*

Years of Assurance.	Expected Deaths by E. L. No. 2.	Percentage of Expected on Lives at Risk.	Lives at Risk.	Actual Deaths.	Percentage of Actual on Lives at Risk.	Percentage of Actual on Expected Deaths, E. L. No. 2.
1st (half of)....	104·103	1·302	7,993·5	54	·675	51·871
2nd	187·242	1·344	13,931·5	142	1·019	75·837
3rd	162·260	1·393	11,641·5	145	1·245	89·362
4th	141·616	1·446	9,789·	117	1·195	82·617
5th	125·684	1·502	8,367·5	87	1·039	69·220
6th	113·586	1·561	7,272·	111	1·526	97·723
7th (half of) ..	50·691	1·621	3,126·5	50	1·599	98·637
First six years..	834·491	1·414	58,995·	656	1·111	78·611
Total after six..	567·921	1·855	30,606·	534	1·744	94·027

It thus appears, that while the Scottish Amicable male lives, non-hazardous, show still a higher percentage of excess of expected deaths during the whole experience and during the first six years, after the first six years the "male lives (town)" show a larger percentage. Thus:—

Table No. VI.

Deaths.	Male Lives (Town).			Scottish Amicable Males Non-hazardous.		
	First Six Years.	After First Six Years.	Total.	First Six Years.	After First Six Years.	Total.
Actual	656·	534·	1190·	322·	310·	632·
Expected by English Life No. 2	834·491	567·921	1402·412	461·334	315·659	776·993
Excess of Expected Deaths	178·491	33·921	212·412	139·334	5·659	144·993
Percentage of Excess on Actual	27·20	6·34	17·84	43·27	1·82	22·93
Percentage of Actual on Expected	78·61	94·02	84·85	69·79	98·20	81·34

I shall be very glad if it should be found that we may anticipate a less rate of mortality amongst select assured lives after six years than the English Life Table No. 2 indicates; but while the "male lives (town)," when thus dissected, do exhibit a more favourable view, yet if we take the Equitable "male lives (town),"

and Scottish Amicable male lives non-hazardous, together, the agreement is very close, and the probability is that there will not be found to be any material error in the assumption, especially for calculating the values of annuities.

The expected deaths after the first six years, by the English Life Table No. 2, are—

For "male lives (town)"	567·9
,, Scottish Amicable non-hazardous males .	315·6
,, Equitable	4,067·3
	4,950·8

While the actual deaths are—

"Male lives (town)"	534
Scottish Amicable non-hazardous males . .	310
Equitable, in one view	3,936
	4,780
Adding to the Equitable in another view, as explained in Number XLVI. of the *Magazine* .	160
	4,940

According to the principles on which Table VI., in former Number was calculated, a correction might also be made on Column No. 3 of that table (percentage of mortality, "town males," first six years of assurances). This would cause a small deduction from the percentages, and, to a very trifling extent, would affect the values of annuities for "town males" under Table VIII. in said paper. It is not, however, worth while making these alterations, especially as the experience founded on gives only calendar years, and not actual years of assurance. The rates of mortality, therefore, for first six years, in any view, can be only approximate, and other approximations might give higher percentages. It is hoped that it will now be apparent that it is very desirable to have experience data so framed as to exhibit the experience of actual years of assurance, and not merely calendar years, especially as many Offices, it is understood, do a much larger business at the close of the year. Even, however, if the business were regularly distributed over the year, it is not satisfactory to deduce the mortality for the first year of assurances, either by taking double the first calendar year or adding to the mortality of the first calendar year half of that for the second year, and so on. In later years such a mode could make little or no difference in results; but, independent of the marvellous difference represented to exist in the London Equitable Experience in

the first and second calendar years of entry (which may probably admit of some explanation), it is sufficiently obvious otherwise that the advancing rate of mortality in the early periods of assurance is so strongly marked as to require at least exact divisions of the years of the duration of assurances to enable the facts to be sufficiently accurately ascertained and exhibited.

Among assurance statistics I have not observed any probable estimate of discontinuances otherwise than by death in each year of the assurances.

The following is a

TABLE (No. VII.) *showing the Percentage of Discontinuances in various Years of Assurances according to the Experience of the "Male Lives (Town)" of the "Experience of Offices" and the Scottish Amicable Male Lives Non-hazardous—the Discontinuances being, in the case of the former, reckoned as Policies discontinued in each calendar year of assurance out of Policies at commencement of year (or at entry in the case of first calendar year), and, in the case of the Scottish Amicable, being reckoned as years of risk discontinued in each year of assurance out of number at risk for the year plus such years of risk discontinued; also Rough Estimates of the probable Discontinuances in each Year of Assurance out of* 100 *at commencement of each year, and out of* 100 *at entry.*

Year of Assurance.	DISCONTINUANCES OUT OF 100 AT COMMENCEMENT OF EACH YEAR.			Roughly Estimated Discontinuances out of 100 at Entry.
	Scottish Amicable Male Lives Non-hazardous.	Town Males.	Roughly Estimated.	
1	1·3	·7	1	1
2	9·8	9·9	10	9·9
3	7·1	6·7	8	7
4	5·5	5·8	6	5
5	5·1	3·7	5	4
6	4·7	3·9	4	3
7	4·1	3·5	3	2
8	4·3	6·0	2·5	1·5
9	2·8	2·3	2	1·3
10	2·3	1·7	1·9	1·1
11	3·7	1·7	1·8	1·0
12	2·7	1·8	1·7	·9
13	1·7	1·4	1·6	·80
14	2·7	1·4	1·5	·74
15	2·6	1·7	1·4	·68
16	1·2	0·2	1·3	·62
17	2·2	0·4	1·2	·56
18	1·2	0·0	1·1	·51
19	1·	0·0	1	·46
20	1·6	3·0	1	·43
				42·50
21 and after	. .	. .	1	7·5
				50·

It will be seen from the preceding table that I estimate that about one quarter of the policies issued are discontinued otherwise than by death in the first five years, and about a like number after that, or one-half of the whole. This includes short period policies which have run their term, but these must be a very small proportion, and the number otherwise discontinued is obviously very large. No doubt the estimate will be high for the experience of some Offices, but taking in all Assurance Offices, I should not consider it an over estimate according to past and present experience. It is to be hoped that, both as an index of the general advance of provident habits and of greater appreciation of the advantages of life assurances, future experience may establish a smaller proportion of discontinuances.

Notes on the Early History of Tontines. By FREDERICK HENDRIKS, *Esq., Actuary to the Globe Insurance Company.*

IN the third of my papers entitled "Contributions to the History of Insurance and of the Theory of Life Contingencies,"* after treating upon the earliest examples which, after a somewhat tedious search, I could find of tontine associations practically carried out—examples confined, as it will be recollected, to some comparatively small projects of the *Dutch* municipalities of Kampen, Middelburg, and Groningen, employed for raising capital sums, by the grant of tontine life annuities, in 1670 and 1671—the circumstance was mentioned, that the credit of the invention was properly accorded to the Italians, or more particularly to the Italian *Tonti,* whose name has been so effectually incorporated into several languages through the word "tontine."

Upon the occasion referred to (writing in 1852), I mentioned that it would be desirable to ascertain whether previous tontine annuity schemes—as, for instance, those instituted by Tonti, in France or elsewhere, and prior in date to 1670—included any tables assimilating to that of the Kampen tontine of the same year.

Although the special object of this note was not stated, it may be mentioned that it was as follows:—The sixth column of the table of the Kampen tontine showed an estimate of the number of persons who, from the age of 1 to 60 years, in each twelve years, and thenceforward for each year up to the age of 80 years, are assumed

* *Vide Assurance Magazine and Journal of the Institute of Actuaries*, vol. iii., pp. 93-120, and particularly § 29, p. 111, and Note, p. 116.

to remain still in existence. The projectors of the tontine stated (with reference to this column) that they are aware that anything like a precisely accurate calculation of the proportion dying out of a given number is not attainable, but "the assumed numbers had been framed upon a new elaborate scrutiny and observation, and there was no ground for assuming that the results to be experienced would range either below or above the anticipated number of survivors." Upon this I suggested in the context, that the idea of a *life table,* arranged in the form of the number surviving to a given age out of a specified number living at any age, *originated,* roughly though it may have been, in these tontine annuity schemes.

Although ten years have elapsed since this suggestion, such researches as have been made hitherto for the particulars of Tonti's scheme have not enabled us to discover them. If they do still exist, it is probably in a manuscript form in some of the French State archives. In the course, however, of these inquiries, details respecting the biography of Tonti have come under my notice which are not only interesting in themselves, as supplying information not obtainable from English sources, but as establishing the fact that it was not until 1689 that any tontine scheme was successfully carried out in France, although it would appear to have been projected before 1648.

The letters of Tonti to Colbert still remaining in the French archives were brought to light for the first time in France in 1852; but buried as they are in the vast mass of ancient administrative correspondence published by the Government, it will be useful to reproduce them in a more easily available form. I have, therefore (see Appendix A) translated them, and also the excellent prefatory remarks of the late M. Depping; and I am not without hope that they may serve to indicate to future writers, who, in biographical or politico-economical collections, have to refer to Tonti, that more may now be said of his career than the meagre notice, which is all that can be usually found respecting it, as to his place of birth having been Naples, and as to the tontine system having been named after him.

In a paper read to the Statistical Society in May, 1857,* I took occasion to point out that the writings of an old and important author on economical science, John Houghton, showed how early in England, upon the introduction of the tontine system, was the appreciation of the sketch of a table of mortality given by Graunt

* "On the Statistics of the British Land Tax Assessment," *vide* Journal of the Society, vol. xx.

in his well-known *Observations upon the Bills of Mortality,* the first edition of which appeared in 1662. Houghton was an apothecary in the city of London, a Fellow of the Royal Society and a contemporary of Halley, Evelyn, Pepys, and Sloane, who all had a high opinion of his talents and contributed to the serial publications which he issued, in twopenny and penny numbers, between Sept., 1681, and Sept., 1703. The mass of these publications appeared under the title of *Collections of Letters for the Improvement of Trade.* Such of them as had appeared in *folio* (omitting the advertisements, the wind and rain tables, the stock and share lists, the weekly prices of corn in various parts of the country, the prices in the provision and other markets, and a condensed notice of special news from the *London Gazette*) were republished, twenty-four years after the date of the last paper, by Richard Bradley, Cambridge Professor of Botany (3 vols. 8vo.; London, 1727). In this republication Bradley notices that he "had the greatest pleasure in the opportunity of restoring to the world Mr. Houghton's useful writings, which, in all probability, would never else have been again published; and" (he continues) "it is very likely that there are not, in all our English libraries, ten complete sets; for, as they were published in single papers, many of them were undoubtedly lost, and the few that are now left are esteemed as valuable as choice manuscripts."

Bradley was either unacquainted with, or else could not obtain, the papers which Houghton had published in a *quarto* form between 1681 and 1693. In one of these papers (No. 13, for Tuesday, Feb. 13, 1683), entitled "Some Considerations upon the Proposals approved of by the City of London for Subscriptions on Lives, wherein are some Observations and Conjectures upon the East India Company and Bankers," the tontine plan is so well described, and the statement it contains of Graunt's Table of Deaths, in decades of age, as applicable to determine the "reasonable" expectations of survivorships, is so important—preceding in date, as it does, the more learned mathematical researches of Dr. Halley on the Breslau Table—that it cannot fail to be interesting to the actuarial as well as the general readers of the *Assurance Magazine.** It is with this view here reproduced from my printed copy. (See Appendix B.)

* The identification, by Houghton, of Petty's labours with Graunt's, where he observes that "Major Graunt, or rather that learned and ingenious virtuoso, Sir William Petty, in his admirable *Observations of the Bills of Mortality* of London, &c. . . This great man saith," &c., may appear to lend some colour to the early origin of the question which Mr. Hodge and Professor De Morgan discussed in this *Journal,* as to which of the two (Graunt or Petty) was the real author of the celebrated *Observations;* but I am inclined to the same conclusions as Professor De Morgan, that Graunt was *the* man, and that Sir William Petty may have very materially assisted him.

Appendix A.

The following notes and correspondence are extracted from *Correspondance Administrative sous le Règne de Louis XIV., entre le Cabinet du Roi, les Sécrétaires d'État, &c.* Recueillie et mise en ordre par G. B. Depping. (Paris; Imprimerie Impériale, 1852; 4to.) See volume iii., "Affaires de Finances, Commerce, Industrie:" Introduction.

M. Depping remarks upon its being evident from Tonti's letters to Colbert, that it was under the ministry of Cardinal Mazarin that Tonti came to Paris and offered his plan of a tontine to the Government. The Cardinal understood, or rather exaggerated to himself the advantages of this financial enterprise, and fancied he perceived in it millions of profit for the State. The tontine was adopted, and the inventor received a pension of 6,000 livres, which proves to what degree the Cardinal's mind had been struck by the importance of the financial ideas of his countryman. The latter, in his letters to Colbert, says that he received this pension from 1648 to 1660; that in the following years he received no more than trifling instalments. From that time his letters are filled with lamentations upon the discomfort in which he was living with a family of seventeen or even nineteen persons. He avows his miserable condition, expresses his fear of being cast into prison by his creditors, and implores the pity of the king and of the minister—conjuring the latter, by the name of Mazarin, their former common protector, to come to his help. These complaints are very earnestly expressed in the years 1663, 1664, and 1665—that is to say, at the very time when the royal edicts confirmed the establishment of the tontine as a State institution. He asks for Madame Colbert and the queen to interest themselves in his daughters '*grandes et bien faittes*,' as he says, in order that they may be admitted into convents. His embarrassments do not prevent him from meditating upon other projects useful to France. He proposes the establishment of a new India Company, which would procure for the State a fund of 40 to 50 millions; he advises plantations of mulberry trees upon all the great roads of the kingdom, so as to dispense the population from having recourse to foreign silks. Then one learns from his correspondence, that he had written, in 1664, a relation of the present conduct of the court of France; that the chancellor, Le Tellier, in the king's name, had prohibited him from publishing this work, which, nevertheless, in spite of himself, so it is asserted, a bookseller caused to be printed.*

We are ignorant of what occurred concerning him during the subsequent years; it is a question whether this work or some other indiscreet act drew down upon him disgrace at court. It is not in my power to answer. Tonti disappears from the scene of this world, and I confess that I have been extremely surprised to find a letter of his dated from the Bastille, 4th March, 1675, and conveying the information that he had been languishing for seven years in that prison with his two sons, and that his

* "Last day of August, 1665. I humbly supplicate your Excellency to obtain for me, from the king, the justice for which I have appealed to His Majesty, against a bookseller who has had the effrontery to cause to be printed the account of the present conduct of the court of France, which I composed last year, and which His Majesty had commanded me, for certain reasons, through M. Le Tellier, not to have printed. I hope for this favour through the goodness of your Excellency." (*Vol. Verts, C.*)

family, from whom he was cut off, was living in the profoundest misery. The unfortunate man does not ask for his liberty; he knows, perhaps, that he will not obtain it; he only implores Colbert, in the name of God, and by his veneration for the memory of Cardinal Mazarin, to continue to have pity upon him, and not to leave those belonging to him without help.*

From this period the traces of his existence vanish. The Bastille has presented more than one mystery of such a kind. It is only known that one of this Tonti's sons became a distinguished officer and signalised himself in the expeditions against the savages in America. It is strange that the historians of Louis XIV. and contemporary memoirs preserve silence respecting the fate of the unfortunate Italian, who probably had more ideas than experience, more dash (*plus d'abandon*) than prudence.

His plan of a tontine was, besides, conceived more in the interest of Government than of private persons investing in it. It was a voluntary loan which he desired to effect through its means, by offering to lenders the temptation of a large future interest. One is forced to believe that the public was not taken by it, and that it was from mistrust that the French—who, in general, at that period, were not at all used to risk their funds in speculations of chance—showed but little eagerness to enter into the tontine association. The first which Tonti proposed, in 1653, was not carried out; the second, which was brought before the public three years subsequently, under the name of the 'Royal Bank', and which concealed a lottery, but wherein the Parisians persisted in seeing another tontine, was, by virtue of royal letters patent, registered, and had a commencement of organisation, but was not supported by the investments of the public. Untired of invention, the Neapolitan economist would soon have carried out another project, that of the ecclesiastical tontine, to furnish funds with the view of extinguishing the debts of the clergy, which was again a loan. No one allowed himself to be taken with it, and as the poor inventor was languishing in prison, there was no further question of tontines. But in the sequel, when the embarrassments of the treasury obliged recourse to all kinds of expedients, the Count de Pontchartrain wished also to try and revive the tontine—that is to say, to procure funds for the State by means of the plan originally conceived by Tonti. In 1689, therefore, there apppeared a royal edict for the organisation of a tontine, in which each person, by depositing a sum of 300 livres, might assure, for his old age, a rent increasing with age. This, at least, was completely enough carried out to encourage the Government, some years after the first edict, to found a second tontine upon the same plan. If it did not have all the success hoped for from it, the two associations contributed at least to diminish a little the distress of the treasury.

The letters referred to by M. Depping are the six following :—

* "Bastille, 4th March, 1675. I return you a million thanks for the aid of 600 livres that your Excellency has obtained for me from the king, which have been used in procuring for me clothes and linen, as also in giving like articles to my two sons, who are confined in this place with me. I hope also that your Excellency will procure for me from His Majesty, when the good God shall inspire him to do so, the 1,600 livres which I owe to the people of this castle, who have furnished me, for seven years, with the things which were necessary for me and likewise for my sons; and nevertheless, Monseigneur, I implore you, in the name of God and by the memory of the late Monseigneur the Cardinal Mazarin, to continue to me your bounty for the subsistence of my daughter, who has the charge of the rest of my family, which is reduced to the last stage of distress." (*Vol. Verts, C.*)

(1) Tonti *to* Colbert.

"*Paris*, 19*th January*, 1663.

"M. Le Tellier has this morning told me that the King had commanded him to return you my petition, similar to the one which is hereto annexed, which I presented to His Majesty a few days ago; and he has ordered me to see you thereon. You will recollect, if you please, Sir, that you had led me to hope for some relief if the matter depended upon you; and now that it is in your hands by order of His Majesty, I very humbly beseech you to do me the favour, when you speak to him about it, to represent to him the deplorable state to which I am reduced, with a numerous family of seventeen persons, by having sacrificed myself for his service, and by the great losses I have undergone through the same cause. And, if my past services are no longer to be considered, I entreat you to submit it to His Majesty that I am on the point of perishing in a prison, and of being denounced; and should this take place it would bring a notable harm to the tontine business, which the late Cardinal esteemed as a mine of gold for the King, and that there might be drawn from it every year several millions which would never be subject to redemption. I hope, by your goodness, and your zeal in the King's service, to receive from His Majesty, through your medium, a prompt help to solace me in my afflictions, and to remain for ever obliged to you for it."

(2) Tonti's Petition to the King.

"Laurens Tonty humbly beseeches your Majesty to consider that he has been, and still is, obliged to go to much expense for the advancement and establishment of the tontine, delayed on account of the Chamber of Justice; and that he has received nothing from your Majesty for the years 1661 and 1662 of the pension of 6,000 livres a year which the late Cardinal Mazarin caused to be given him, by order of your Majesty, in consideration of the great services he had rendered to your Majesty, both within and without the kingdom. And petitioner finding himself burdened with a family of nineteen persons, and, amongst others, with five grown up and well favoured daughters, it is altogether impossible for him to be able to subsist any longer if your Majesty should not have the goodness to order the said pension of 6,000 livres to be paid up for the past years. M. du Plessis Guénégaud, Secretary of State, can testify to your Majesty that he has issued, by order of His Eminence, every year, down to the year 1660, the ordinances for payment of the current instalments of the said sum of 6,000 livres, which were paid to him by the treasurers of the savings fund. And will, in duty bound, pray God for your Majesty's health and prosperity."

(3) Tonti *to* Colbert.

"1*st March*, 1663.

"I return you a million thanks for your goodness in causing the second edict for the tontine to be sealed; and I hope that, through your ministry, the King will soon see this business established, to draw from it the great advantages which may be expected immediately and prospectively. Following your advice, I have spoken to the King, supplicating him very humbly to have mercy upon me, by having continued to me the help which the late Cardinal accorded me, as much in consideration of the services I had rendered to the State, and the loss of property I have endured in

sacrificing myself for the crown, as of the judgment which His Eminence had formed upon the tontine affair as a hidden treasure in this kingdom. But I hope everything from your protection; and as you have graciously told me that this affair (the tontine) is a good one, I conjure you, in God's name, and by the memory of His Eminence, to inspire the King with the same sentiments respecting it as you have; and, at the same time, to do me the service of representing to him the trouble in which I am, with a family of nineteen persons, as also that I am continually forced to be at expense and care in perfecting this affair, for which I am not in a competent condition without the help of His Majesty, and which I entreat you, Sir, very humbly, to procure for me, and shall for ever be obliged."

(4) TONTI *to* COLBERT.

"19*th October*, 1663.

"The Marshal de la Meilleraie had decided upon the tontine affair as extremely good and very advantageous for the province of Brittany, but considered that you could not wait for the issue of the affair, seeing that the States would have to pay every month a portion of the money for which they were bound for the King's buildings, and that another fund would have to be created for the tontine, which is impossible for them at present, on account of the engagements which the said province has been obliged to undertake, finding the whole of its means alienated for four or five years. That which consoles me for not having been able to succeed in that country is the order which the King has caused to be conveyed to me through M. de Lionne, not to send this (tontine) affair to Rome, where it would have infallibly succeeded, to the prejudice of His Majesty; and, if it had been put into operation in Brittany, would by this time be published throughout the country.

"I took the liberty, within the last few days, to give you occasion to beg Madame, your wife, to be good enough to contribute, by her interest and endeavours, to give two of my daughters as spouses to Jesus Christ, and they will pray all their lives for the health and prosperity of all your family.

"I have submitted a petition to the King, humbly begging him to consider, that for three years I have only received 3,000 livres of the pension of 6,000 livres a year which His Majesty had caused to be paid me from the year 1649 down to 1660, in consideration of my services; and as I am pursued by my creditors, and am bound to give honourable subsistence to my family of seventeen persons, according to my position, I have had recourse to His Majesty to receive of his goodness the wherewithal to remedy my present necessities. I very humbly entreat you to support it with your protection, and to continue to me your favours, which will secure my lasting obligation."

(Accompanying this letter is a note addressed by Tonti in favour of his daughters, "*grandes et bien faittes*," for the placing of whom in a convent he wishes that the Queen would interest herself.)

After the lapse of a year, Tonti was again driven to solicit payment of his pension. He wrote to Colbert on the 27th November, 1664:—"You will remove your favouring protection from a family of condition which is very numerous, and for whom it is impossible to find further means of

subsistence, as it is on the point of seeing me perish in prison, if your Excellency does not procure it some prompt assistance from His Majesty out of what is due to me for the past, and by the restoration of the pension which has been paid to me from the year 1648 down to 1660. I conjure this most pressingly, by the glorious memory of his late lordship the Cardinal Mazarin, by your own individual generosity, and by the respectful submission with which I am," &c.

(5) Tonti *to* Colbert.

"It has occurred to me, that, after the approbation with which your Excellency has honoured the tontine, and after the evidences which you have given of wishing for its establishment, your Excellency would not disapprove of the liberty which I take to disclose to you an expedient which has come into my mind, and which I submit to your judgment and good pleasure in the contingency of the redemption of the rents of the Hotel de Ville. The majority of those who will have received their money will experience difficulty in investing it safely and usefully—being possessed, perhaps, of that alone for their subsistence—and will be rejoiced to place it advantageously. To which end it has seemed to me that the establishment of the tontine will be most appropriate, by giving, through His Majesty, to those who voluntarily consent to it, a quarter more in places in the tontine for the whole of their redemption, or for a part of it, of that which it shall be fixed to give them in ready money at their own choice and liberty. Wherein the advantage would be reciprocal, inasmuch as, instead of a small fixed sum which the proprietors of the said rents accepting the said places would receive, after many difficulties raised by the officials to delay the payment, they would have their revenue at 7·14 per cent. (*au denier* 14), and would see it increase every year. And in order to compensate for the loss of their funds in case of death, they would put themselves into a position, by this voluntary acceptance of places for a whole or part of their redemption money, to enjoy, previous to their death, sums to the extent of 100,000 livres of revenue, which would exceed not only the principal, but also the rent of their money in some traffic or legitimate trade in which they might employ it; and the great advantage to His Majesty would be, that, without opening his purse, he would inherit the revenues of each class by the death of the last survivor in it, and would thereby find himself relieved from payment of the interest of the places without redeeming the funds.

"It is for your prudence, my Lord, to decide this, and for the perfect knowledge which your Excellency has of the equity or possibility of things and of the proper time to put them into practice; the introduction only of the subject have I thought it open to me to make with the submission which is due," &c. &c.

(6) Tonti *to* Colbert.

"22*nd May*, 1666.

. . . . "I believe there is an easy means whereby to procure fresh funds of 40 to 50 millions. It should be managed so that all persons in the kingdom—nobles, officers, and others—will ask to subscribe, and with joy, even when they might be sure to lose their investment, by reason of the great advantages they would otherwise derive. If the whole of this great fund cannot be at once employed in commerce, the King might make

use of a part to redeem his domain, and for the execution of other designs. This might be done without its being known; and should it be known, no complaint against it would be brought forward, as interest would be paid to the Society.* By the same means everyone would speak well of the Society, and would procure its preservation and augmentation, instead of as at present, everyone speaking ill of it. Officers particularly complain that they were constrained to enter it—that it is a trap to catch the money of those who have invested in it. To shut the mouths of the dissatisfied, if your Excellency judge it to be expedient, after this great fund shall have been created, one might make an offer to all those who had already entered the Society, to return them their money if anyone regretted having done so. Not a single person would withdraw, because of the great advantages he would derive through my expedient. If the proposition be agreeable to your Excellency, I will also present the methods of showing the facility with which it may be carried out."

Appendix B.

A Collection of Letters for the Improvement of Husbandry and Trade.

Numb. 13.—Tuesday, February 13, 168⅔.

The contents:—Some Considerations upon the Proposals approved on by the City of London for Subscriptions upon Lives, wherein are some Observations and Conjectures upon the East India Company and Bankers.

Sir,—In my last I gave you an account of the great undertaking, to promote trade by a bank, approved on by the City of London. Now I must give you an account of another great affair, carryed on by the same persons, and 'tis printed in a large sheet by them as followeth, viz.:—

"Proposals made and approved by the City of London for a yearly increase of wealth, by subscriptions, to advance money at interest, for lives, of whatsoever age or sex, under ten several ranks or classes; which subscriptions will produce great advantage to the survivours, as is particularly instanced in the schemes and paragraphs following, viz.:—

	The particular Summs to be subscribed.	The Number of Persons requisite for each Rank.	Total Summe to be Subscribed by each Rank.	The Yearly Interest thereof.
1	5	2,000	10,000	600
2	10	1,000	10,000	600
3	20	500	10,000	600
4	25	400	10,000	600
5	50	200	10,000	600
6	100	100	10,000	600
7	200	50	10,000	600
8	400	25	10,000	600
9	500	20	10,000	600
10	1,000	10	10,000	600

* M. Depping states that this has reference to the organisation of a grand India Company, a project upon which Tonti does not give further explanation.

The Subscribers' Names, Qualities, Sexes, and Places of Habitation.	1. Rank of Persons under 7.	2nd. Between 7 and 14.	3rd. Between 14 and 21.	4th. Between 21 and 28.	5th. Between 28 and 35.	6th. Between 35 and 42.	7th. Between 42 and 49.	8th. Between 49 and 56.	9th. Between 56 and 63.	10th. Between 63 and 70.
A. B., of Cheapside, London, Goldsmith, for his son John....	5									
The same, for C. D., of Cornhill, Gent.	..	..	..	5						
E. F., of Paternoster Row, Mercer, for himself	..	..	..	..	..	..	..	5		

"By which schemes it appears there is £10,000 to be subscribed on each rank or class of summs and persons, the interest of which comes to £600 per annum. But no subscriber shall be held obliged to pay his money until the said summ of £10,000 be fully underwritten—(that is to say) if he have subscribed £5, until £1,999 other persons of the same rank or class (as to age) have each of them subscribed £5 to make up the same £10,000; or, if he have subscribed £50, until 199 other persons of the same rank or class (as to age) have each of them subscribed £50.

"Every person for whom a subscription shall be made, shall, from the time his money is paid, receive interest after the rate of six pounds per cent. per annum; and as any of those that are in the same rank dye, the survivours of that rank shall receive the interest money that should have been paid to the deceased, equally divided amongst them: (that is to say) if one of the ten subscribers for £1,000 each man, do dye, the other nine that shall survive shall receive, besides the interest for their own £1,000 (each) subscribed (which is £60 per annum), their equal share of that £60 which would have accrued to the deceased, and so of the rest.

"Every subscriber, when he comes to subscribe, shall declare the age or ages of the persons for whom he doth subscribe—(viz.) whether they be under the age of 7 or above 7, and under the age of 14 or above 14, and under the age of 21 or above 21; and so of the rest of the said ranks. And if any person underwriting, either for himself or any other, shall declare and subscribe the person for whom such subscriptions shall be made to be of any other age than by the rules or instances aforementioned are allowable to pass in such rank or class as he shall underwrite for, such subscriptions, and the moneys thereupon paid, shall be forfeited, and go to the rest of that rank.

"If any person shall underwrite the first five summs above mentioned, which amounts to but £110, and enter himself, or those for whom he underwrites, according to their respective ages, in five several ranks or classes, he hath thereby a possibility of receiving five times £600, which is £3,000 per annum, during his life, if he survive the rest. And so if any shall underwrite (in like manner) all the ten summs first abovementioned, which amounts to £2,310, he hath like possibility of receiving ten times £600 per annum during his life, if he survive the rest.

"Every person, when he comes to demand any interest money, must bring certificate from the trustees of the respective rank or class in which

the subscription was made, that the person upon whose life the money is demanded was alive at the time when the interest by him demanded became due; but if no demand be made of any person's interest by the space of three years successively, such persons shall be held as dead, and his interest for the said three years, and until he shall afterwards appear to make demand, shall be lost to him, and divided among the rest of the subscribers; which said trustees shall, for the first year, be nominated and chosen by the major part of the subscribers in each rank respectively, as soon as the said rank or class shall be filled up, and afterwards annually.

"If any person desires to transfer his interest, it will sell every year for more money than it would have done the year before; for, the more persons dye, the greater will the income be to the survivors; and any man may transfer his interest to whom he pleases.

"This fund, for the security of the payment of the said interest, shall be settled to the satisfaction of the subscribers, as shall be advised by council learned in the law."

But to these proposals I hear two great objections—

1st. That the subscribers shall have but common interest, and loose the principal; which is not a pennyworth for a penny.

2ndly. That people may club together, and leave to the longest liver interest and principal; or it may be again divided among the heirs, or the heirs may come in afresh upon interest, and it may be a fund for such interest *ad infinitum.*

These objections I'll consider with all fairness imaginable; and first, of the first.

'Tis true the subscribers shall have no more but what the law calls common interest—viz., 6 per cent; but whether the currant interest of the town and country is like to be such, is to be enquired into; and so much as the interest shall be less worth, so much will the proposals appear better; and if my foresight fails me not, within a while interest will not be so much worth. My reasons take as follow:—

Within this two years I went myself to the East India Company, in behalf of a friend, to offer them some money at 3 per cent.; and though I made one of the officers my friend, to entreat for me, yet it would not be accepted. And 'tis notorious that, about that time, abundance of people did lend them at that rate; the reason, I suppose, was, because they could not get more with security to their satisfaction. Nay, to others 'twas lent at 4, 4½, and 5, currantly; and although now 'tis risen to 6, yet when the cause of its rise shall be removed, it must of necessity again fall (for it is out of the power of laws to ascertain interest, as is apparent by these instances, and the high rates the King and others have given when their extraordinary occasions have forc't it, besides procuration, continuation, &c.), and the causes of its rise, according to my best observation, have been as follows:—

Besides the great quantities of money carryed out by the Turky Company, and to several other places in Europe, according to common custom, several interlopers for India provided themselves with a great quantity. The East India Company hoping, by a very great trade, to prejudice these interlopers, provided much more than ever they did before. These extraordinary occasions, unless the quantities were proportionably increast,

could do no less than raise interest (every one raising their commodity according to the eagerness and multiplicity of good chapmen); but that which made it more than ordinarily break out was, that some persons that were not pleased with Sir John More's government in his mayoralty, thought to prejudice him by draining Ben. Hinton, his intimate, a goldsmith and banker in Lombard Street. This was told to me about a fortnight before the first bankers of this late storm—viz., Mr. Addis and his partners—went off; and this is confirmed to me by a considerable man in this city, who tells me that Dissenters say the bank will incourage trade too much to be countenanc'd at this time. When money was thus drawn from Mr. Hinton, almost everybody (although 'tis probable the most part knew not why) thought it best to secure their own, and ran with open mouth upon all the bankers for money, thinking it better to let it lye dead a while in their chests, than to run a hazard of trusting such, who, for ought they knew, might do as Mr. Addis and some others near him had done. To joyn with this, some Dissenters being excommunicated, and a discourse that all the rest that would not comply should, made, I suppose, must folk willing to have their money out of such hands. Also the many rich interlopers that went and were designed to go to India, together with all the jealousies imaginable raised by them and their friends upon the Company, made a great many of the fearful members of the Company eagerly sell their stocks; and, perhaps, some of their designing ones too, that they might afterwards have opportunity of buying again cheaper.

This, when the Company had most need of money to set out their numerous fleet—upwards of thirty sail—made their creditors run very earnestly on them also for the money they had lent them, which put them to such straits, that, instead of 3, they were willing to give 5 or 6 per cent., and some say promises of good turns into the bargain. But all would not do, their auctions fell from 365 to 245, and they were forc't to put a stop to payments for three months; and, in the meanwhile, have appointed a sale, and expect several rich ships home, with which they question not to give a stop to all reasonable complaints.

This, all laid together, I take to be the reason why money is at this high rate; but if my conjectures are false, I beg pardon, and I wish some more knowing would give us the true causes. But if I am not mistaken, then it will follow that when the hoarders are weary of keeping up their money, when what was exported last year to all parts shall return with a duck in its mouth, when the East India auctions shall again rise, when the fright shall be over (as usually in these cases it is after a little while), when we shall come to have less disputes and be more united about religion—all which I hope shortly to see—then you'l find money as cheap, if not cheaper, than usually, and the East India Company offered again more money than they have occasion for; and I verily believe they may, if they will, be one of the chiefest funds in the nation; for although they owe a great deal of money, yet 'tis visible that they have a far greater stock; and 'tis also plain that 'tis their interest to keep up their credit, although it were by lending their own private cash, for otherwise they loose more by the fall of their auctions then all their debts come to; as lately 'twas said, their debts were about £800,000, and their principal stock is about £750,000, and every £100 fell in their auctions, as above, from 365 to 245, which is 120 per cent., amounting to £900,000.

This money, or so much of it as will make currant payment, I perswade myself they will raise by themselves and friends, unless they find some better expedient, although, perhaps, forbearance of a dividend or two may do the business. If so, then, seeing the bankers are single and their stocks not so visible, and some or other of them drop off, what should hinder but (this Company appearing thus staunch) most folk should run their money in here cheaper than other places by 1 or 2 per cent. (except in the Guinia Company, which I take to be as safe as this). And as for dangers from interlopers, I see no great reason to fear, because the Guinia Company, in spite of as many interlopers as will go, are in a thriving condition. And this India Company may, with a less gain per cent. in many more hundreds, get more money and more dishearten their adversaries. There is no necessity for a double trade, to have double fortifications, double agents or factories, neither will their need to the great ones in India double presents, nor altogether be among themselves double petty charges. Much more I could say on this subject, but I don't think it needful here; my drift is mainly to show that if money should come again to 3 per cent., then the allowance of 6 will be equivalent to 12 when interest is at 6.

The second part of the first objection is, that the principal will be lost.

'Tis true it will be so, but who is it will find it? Why, 'tis the City of London, from whom most of the money expected to be subscribed hath been gotten; for if they are citizens they give it to their own body and for their own use. 'Tis probable their children may be the orphans to reap the benefit of it. But suppose it should be spent in triumphs, Lord Mayor's shows, publick buildings, festivals: is it not such like that inables us now to live so well, and makes us the renown of the whole earth? I am strongly perswaded that a stately Lord Mayor's shew makes London, from strangers that flock to see, get more money, six times over, than ever the charge of the pageantry came to; and if they had stock to enlarge their glory, I am sure a proportional expense will follow it. O what crouds flockt hither to see the glory of the King's return and coronation.

But if the subscribers should not be citizens, yet it is likely it will turn to their children's advantage; for London is the means of preferment to most of the country's progenie. And I question not, but when the city shall reap these advantages, 'twill be a good argument for preferment to places for the heirs of such subscribers.

To the second objection—viz., that the people may club, &c.

I confess, in theory, all this is true; but 'tis next to impossible to believe that e're it should be practised; for who, without some consideration of loaves for his pain, will gather this club together? or will they *more fungorum*, as mushrooms, all start up together in a night to throw in their dust? But if they should, who should find out a security, search the title, or be council learned in the law, to settle and secure this fund? Or, if paid for, will not the charge of doing these things, when taken out of these little combinations, reduce this profit to a less than what is here proposed? I doubt it will. However, till that be tryed, this is the better; and everyone hath liberty to please himself. I believe I shall never be of that club; whatever I shall be of the other, I can't yet say. If the first part of this objection won't take, the rest ne're can, as being dependants on it.

Hoping that what's here said may be some answer to the objections—or, at leastwise, make the prejudice of subscribing appear not altogether so great as at first it might be thought for—I'll strive now to show what in likelihood will be the advantages of these subscriptions.

'Tis told you 6 per cent.—with the whole, £600 per annum—to be divided among the survivours, even to the last man. Nobody subscribing for himself that looses here, but he that dyes; and rather than he shall want money in the next world, he, if he leads a good life here, shall have a note to St. Peter to turn the key for nothing. But for those that shall live long here, let's see what in likelihood shall be their profit.

Major Graunt, or rather that learned and ingenious virtuoso Sir William Petty, in his admirable *Observations of the Bills of Mortality* of London—a book useful to a multitude of purposes, and a pattern for many other great designs; this great man, I say, in p. 14 of his second edition, saith, "That about one-third of all that were ever quick dye under five years old, and about 36 per cent. under six."

If so, 'tis to be supposed that those of the first rank that shall live but seven years after subscription shall receive upwards of 9 per cent.

But in p. 56, "That three dye yearly out of eleven families, of each eight persons—*i. e.*, eighty-eight; if it were ninety, 'twould be one in thirty. And at this rate, whosoever lives thirty year in a place may have no neighbour that was cotemporary with him in his first year. Therefore, they of the said rank may, in thirty years, reasonably expect almost £200 per annum a piece.

In p. 58 he saith, Of one hundred there dies within the first six years 36; the next ten years,

Or decad	24
The second decad	15
The third	9
The fourth	6
The next	4
Next	3
Next	2
Next	1

From whence it follows, that, of the said 100 conceived, there remain alive

At six years end	64
At sixteen	40
At twenty-six	25
At thirty-six	16
At forty-six	10
At fifty-six	6
At sixty	3
At seventy-six	1
At eighty	0

In p. 65 he saith, In the country about one in fifty dye yearly, but in London about one in thirty; and that London is not so healthful now as heretofore. Wherefore it may be advantageous to subscribe on countrey lives rather than city ones.

If what I have already said shall appear reasonable, then it may be worth while to consider what people in likelihood it may be fittest for.

I do suppose it very proper for all landed men to put in £5 a piece for their younger children, for if they die quickly the estate will be free to the heir; if not, 'twill be considerable, and the estate shall not need to be clog'd for maintenance.

'Twill be proper for all tradesmen who live gentilely from hand to mouth, but never provide much aforehand; this way their children may have portions, and themselves be well kept in their old age.

'Twill make all sorts of old people be made much on, because the longer they live the more they'l have.

Merchants, several other tradesmen, and gamesters, that live by hazard, may sometimes, at extraordinary hits, put something in here to keep in cases of extraordinary losses.

Poor servants, as soon as they get £5, may by it take care for old age.

Friends may put in for their she relations who are ill marryed, whereby they shall never want; and 'twill be a means for forcing kindness from their husbands.

'Twill be better for wives than joynters, and husbands may imploy a great deal of the money that should buy land.

If families grow so low, that there is a necessity for selling of joyntures, some of the money may be secured here, and the widdow fare ne're the worse.

If every University man, when he comes to preferment of £100 per annum, would put in £5, and so for every £100, to have the interest go towards the library of his Colledge, 'twould, in a while, make them very great; and 'twould do the like for Sion Colledge, if each such London minister would do so.

And if twenty good folks would give £5 a piece, for ought I know it might raise such a Colledge as Mr. Abraham Cowley speaks on in his *Discourse of Agriculture for the Improvement of Husbandry.*

Multitudes of other conveniences I could enumerate, but they that will consider these may find enough more that will be agreeable to their own circumstances; and there is a book signed by Mr. Wagstaff, the City Town Clerk, which book is entituled *Proposals for Increase of Wealth by Subscriptions.* That will show you divers other instances.

Whatever is said of £5 will serve for any other summe.

Sir, if these reasons shall be undenyable, I pray incourage this design; if not, persuade somebody to show the contrary. But however, pardon me, who, unless I do subscribe, am like no ways to be concerned in it, except in good wishes for that city from which I have had my well-being.

Farewell, &c.

On the Construction and Use of Commutation Tables for Calculating the Values of Benefits depending on Life Contingencies. By PETER GRAY, *Esq.*, F.R.A.S.*

(Concluded from page 108.)

WE have now to give some examples of compound benefits, which are those consisting of two or more simple benefits; but the combinations which may be formed of these being obviously very numerous, it would be beside our present purpose to attempt giving a complete list of them. Our object will be, in selecting a few of them for illustration, to indicate the method of dealing with the more complicated cases, and also to prepare the way for the most general application of the Commutation Tables, which application will form the subject of the concluding portion of this paper. A very complete list of the formulæ for the more elementary of these benefits is contained in Professor De Morgan's first paper on the subject; and as it is hoped that little difficulty will be experienced with these, after the illustrations to which our space limits us, we shall not scruple, as we have occasion, in the solution of any of the problems with which the present paper will be occupied, to refer to any of the learned gentleman's formulæ which we may not have deduced for ourselves. Our references will be made in the following manner, which is rendered necessary in consequence of the formulæ not forming one consecutive series. Formula 10, on page 16, for example, will be denoted thus, [16, 10]; formula 7_2, on page 18, thus, [18, 7_2]; and so on.*

As we are no longer to confine ourselves to benefits whose amount is £1, we again point attention to a remark made on page 99, to the effect that, when we have the present value of a benefit of £1, that of a like benefit of any other amount will be found by multiplying the first-named present value by the number of pounds in the amount in question. As it is convenient to have distinctive

* Extracted from the *Mechanics' Magazine* for 1842.

† It may be of use here to point out a few typographical errors in Professor De Morgan's papers, which might otherwise embarrass the student:—

First paper, page 11, line 22, for $(A+\overline{n-1}h)$, read $(A+\overline{n-1}H)$.
" " 16, " 29, for $(a+(n-1)h$, read $a \pm (n-1)h$.
" " 17, " last, for SC_{x-1}, read SC_{x+k-1}.
" " 21, " 30, for M_x, read R_x.
Second paper, " 2, " 25, for $(1-v)N_{x,y}$, read $(1-v)N_{x-1,y-1}$.

Also, the terminating braces are omitted in the expressions [18, 11] and [19, 15_2] of the first paper.

symbols to represent the amounts of benefits of different kinds, we make use of the following for this purpose :—

s, The amount of an endowment.
a, The annual rent of an annuity.
h, The annual increment or decrement of a variable annuity.
S, The amount of an endowment assurance.
A, The amount of an assurance.
H, The annual increment or decrement of a variable assurance.

Perhaps we may require a few others. If so, we shall explain them as they are introduced.

Also, since the expression for the present value of a compound benefit is the sum of the expressions for the present values of the simple benefits of which it is composed, and since these are fractions having for their denominator D_x, it will likewise *generally* be a fraction having the same denominator. We shall, therefore, to economise space, usually omit this common denominator; but it must be carefully remembered that the expressions are incomplete without it. We have said that the expression for the present value of a compound benefit is *generally* of the form alluded to. The exception is (De Morgan, I., pp. 14, 15), when a part of the benefit depends on the unknown item of payment. In this case the expression takes another form. When it does so, it will be exhibited without abbreviation.

In what follows we shall no longer adhere to the formality of problems. For after reference, however, we shall number the expressions we deduce with Roman numerals, in continuation of the number at which we have arrived in the previous problems.

Referring to remark 4, on page 178, we further premise, that, benefits being divisible into the two classes of annuity benefits and assurance benefits, if we deduce the expression for a benefit belonging to one of these classes, it will obviously be unnecessary to do so for the corresponding benefit belonging to the other class, since the relation indicated in the remark quoted *always subsists.*

We proceed now to the more legitimate subject of this portion of our paper.

The increasing benefits of which we have hitherto spoken are those in which the annual increase is equal to the first payment. But the Commutation Tables can also be applied to finding the value of increasing benefits, in which the annual increase is in no way dependent on the first payment; and also of decreasing benefits, with the like latitude as to the magnitude of the decrease.

Thus, a life annuity whose successive payments are to be £a, £$(a+h)$, £$(a+2h)$, £$(a+3h)$, &c., may be decomposed into the following annuities, viz.—a life annuity of £a, and an annually increasing annuity, to be entered upon one year hence, of £h, £$2h$, £$3h$, &c. The present value of the first is (Prob. II.), $a\text{N}_x$; and of the second (Prob. VII.), $h\text{S}_{x+1}$. The present value of the compound benefit, therefore, is $a\text{N}_x+h\text{S}_{x+1}$. In like manner it may be shown that the present value of a life annuity, whose successive payments are to be £a, £$(a-h)$, £$(a-2h)$, &c., is $a\text{N}_x-h\text{S}_{x+1}$. The following formula will, therefore, include both cases, the upper sign having reference to the increasing, and the lower to the decreasing benefit:—

$$a\text{N}_x \pm h\text{S}_{x+1} \quad . \quad . \quad . \quad \text{(XIX.)}$$

By the remark on page 178, already referred to, the formula for the corresponding assurance benefits will be

$$\text{AM}_x \pm \text{HR}_{x+1} \quad . \quad . \quad . \quad \text{(XIX.)}$$

According to what has been said above, we shall not usually give the formulæ for the two classes of benefits, since, as we have seen, the formulæ for the one class are so readily derived from those for the other.

If, in (XIX.), $a=h$ for the increasing benefit—that is, if the annual increase be equal to the first payment—the formula becomes

$$a\text{N}_x+a\text{S}_{x+1}=a(\text{N}_x+\text{S}_{x+1})=a\text{S}_x,$$

by (10), which agrees with (VI.), as it ought to do.

While the above formula expresses, for every value of a and h, the true values of the benefit, yet it must be observed, that, in the case of the decreasing benefit, h may be taken so large, that the annuitant (we confine our remark to the annuity, although it is equally applicable to the assurance benefit), if he live long enough, will have to pay instead of to receive. Thus, if a person aged 30 enter upon a decreasing life annuity, whose successive payments are to be £10, £9, £8, &c. (that is, $a=10$, $h=1$), it is evident that the 10th payment will be £1, and the 11th 0. And, since the annuity is for the whole life, the decrease still goes on, so that the 12th, 13th, 14th, &c., payments will be $-$£1, $-$£2, $-$£3, &c.; that is, the annuitant, instead of having anything to receive, will have these sums to pay.

It may be, also, that the present value of the payments to be thus made by the annuitant will exceed that of the payments he

will have previously received. This is indicated in the application of the formula to any particular case, by its numerical value in that case becoming negative, which will evidently be when hS_{x+1} is greater than aN_x. A negative value presented by the formula indicates that the *purchase-money* for the benefit must be paid by the *seller*.

To avoid the inconvenience of the payments becoming negative, h must never be taken larger than $\frac{a}{n}$, n being any number not less than the number of years during which the annuity is to last. In the case of an annuity for the whole life, the least value of n will be the difference between the age at which the annuity is entered upon and the oldest age in the table, when the last payment that can possibly be received will be £h.

If, in (XIX.), we write $x+n$ for x, we are furnished, as in the case of the simple benefits, with the expression for the same benefit to be entered upon n years hence. This expression is

$$aN_{x+n} \pm hS_{x+n+1} \quad . \quad . \quad . \quad \text{(XX.)}$$

It may also easily be deduced by decomposing the compound benefit.

The expression for the same benefit to last n years is deduced in the following manner:—This modification of the benefit consists of an annuity of £a, to be entered upon immediately and to last n years, and of an increasing annuity of £h, £$2h$, £$3h$, &c., to be entered upon 1 year hence and to last $n-1$ years, and which is to be either added to or subtracted from the other, according as the benefit whose value is sought is an increasing or a decreasing one. The present value of the first portion is (IV.), $a(N_x - N_{x+n})$; and that of the second portion is found as follows:—the present value of an increasing annuity of £h, £$2h$, £$3h$, &c., to be entered upon in k years and to last n years, is, by the table on page 177, $h(S_{x+k} - S_{x+k+n} - nN_{x+k+n})$. If, therefore, in this expression, we substitute 1 for k, and $n-1$ for n, we shall adapt it to our present purpose. Making these substitutions, the expression becomes $h(S_{x+1} - S_{x+n} - (n-1)N_{x+n})$. But $(n-1)N_{x+n} = nN_{x+n} - N_{x+n}$; hence the expression becomes $h(S_{x+1} + N_{x+n} - S_{x+n} - nN_{x+n})$. But, by (10), $N_{x+n} - S_{x+n} = -S_{x+n+1}$. Therefore the expression becomes, finally, $h(S_{x+1} - S_{x+n+1} - nN_{x+n})$. And if to this we connect by the proper sign the expression for the present value of the first portion of the benefit, we have, as the expression sought—

$$a(N_x - N_{x+n}) \pm h(S_{x+1} - S_{x+n+1} - nN_{x+n}) \quad . \quad . \quad \text{(XXI.)}$$

This expression may also be deduced in a somewhat different manner, which, as our object is illustration, we likewise insert. The benefit under consideration evidently admits of decomposition into the two following simpler benefits, viz. (confining ourselves, for perspicuity, to the increasing benefit), a uniform annuity of £$(a-h)$ for n years, and an increasing annuity of £h, £$2h$, &c., also for n years, and both to be entered upon immediately. The present value of the first is, (IV.), $(a-h)(\mathrm{N}_x-\mathrm{N}_{x+n})$; and of the second, (VIII.), $h(\mathrm{S}_x-\mathrm{S}_{x+n}-n\mathrm{N}_{x+n})$. Adding these expressions, we have—

$$(a-h)(\mathrm{N}_x-\mathrm{N}_{x+n})+h(\mathrm{S}_x-\mathrm{S}_{x+n}-n\mathrm{N}_{x+n})=$$

$$a(\mathrm{N}_x-\mathrm{N}_{x+n})-h(\mathrm{N}_x-\mathrm{N}_{x+n})+h(\mathrm{S}_x-\mathrm{S}_{x+n}-n\mathrm{N}_{x+n})=$$

$$a(\mathrm{N}_x-\mathrm{N}_{x+n})+h(\mathrm{S}_x-\mathrm{N}_x-\mathrm{S}_{x+n}+\mathrm{N}_{x+n}-n\mathrm{N}_{x+n}).$$

Now, by (10),

$$\mathrm{S}_x-\mathrm{N}_x=\mathrm{S}_{x+1}, \text{ and } \mathrm{S}_{x+n}+\mathrm{N}_{x+n}=\mathrm{S}_{x+n+1};$$

hence the expression becomes, as before—

$$a(\mathrm{N}_x-\mathrm{N}_{x+n})+h(\mathrm{S}_{x+1}-\mathrm{S}_{x+n+1}-n\mathrm{N}_{x+n}).$$

It will be seen, on comparing this expression with (XIX.), that it does not follow the same law as the simple benefits, in passing from the expression for the value of a benefit to last the whole life to that for the value of the same benefit to last n years. Did the law referred to hold here, the signature of N, in the coefficient of h, would be $(x+n+1)$. (See De Morgan, I., page 21.)

We shall not seek to deduce here any more of Professor De Morgan's formulæ. We leave the others as a most improving exercise for the student, and pass on to the consideration of a few miscellaneous benefits.

£A are to be received by (x), or his representatives, at the end of n years, if he be then alive, or at the end of the year in which he dies, if that event take place before the expiry of the n years. Required the present value of the benefit. This benefit is evidently equivalent to an endowment of £A payable in n years, and a temporary assurance of the same amount, to last n years. Its present value, therefore, is, by (I.) and (XIII.),

$$\mathrm{A}(\mathrm{D}_x+\mathrm{M}_x-\mathrm{M}_{x+n}) \quad . \quad . \quad . \quad (\mathrm{XXII.})$$

This is a benefit of not unfrequent occurrence in practice. Several of the Companies publish tables of the equivalent annual premiums, the method of finding which will be shown hereafter.

Required the present value of a life assurance of £A on (x), with which the sum paid is to be returned.

If we call the present value of this benefit P, then £(A+P) will be the sum to be received at death; the benefit, therefore, will be a life assurance of £(A+P), the present value of which is (using the denominator in this case)—

$$\frac{(A+P)M_x}{D_x}.$$

And, by condition, we have

$$P=\frac{(A+P)M_x}{D_x} \quad \therefore\ PD_x=AM_x+PM_x.$$

Whence,

$$P=\frac{AM_x}{D_x-M_x} \quad . \quad . \quad . \qquad \text{(XXIII.)}$$

If, at death, along with the sum assured, the sum paid is to be returned, with simple interest upon it from the date of payment, the sum to be received, if death take place in the first year, will be $A+(1+r)P$; if in the second, $A+(1+2r)P$; if in the third, $A+(1+3r)P$, &c.; r denoting the interest of £1 for a year. The benefit, therefore, is a life assurance of $A+(1+r)P$, increasing annually by rP; and the expression for the present value will be, by (XIX.),

$$\frac{\{A+(1+r)P\}M_x+rPR_{x+1}}{D_x}.$$

Equating this to P, as in the last case, we should find—

$$P=\frac{AM_x}{D_x-(1+r)M_x-rR_{x+1}} \quad . \quad . \qquad \text{(XXIV.)}$$

This and the preceding case illustrate a remark previously made, that when a portion of the benefit depends on the unknown (that is, unknown till the equation is solved) item of payment, the denominator of the expression for the present value is no longer restricted to D_x. These and such like cases, however, belong more properly to that portion of the subject which will be treated in the remainder of this paper.

The annual rent of an annuity upon (x) is to be a for the first k years, b for the next m years, c for the next n years, and d for the remainder of life. Required its present value.

Column S will not serve our turn in this case, since that column is applicable only when the variation in the payments is annual, and equable throughout the whole duration of the annuity.

The problem, as proposed, has no such limitations; and we must, therefore, find the present values of the separate portions of the annuity, and add them together for the whole present value required.

The annuity consists of four portions, the present value of the first of which is, by (IV.),

$$a(N_x - N_{x+k});$$

of the second, by (V.),

$$b(N_{x+k} - N_{x+k+m});$$

of the third, by (V.),

$$c(N_{x+k+m} - N_{x+k+m+n});$$

and of the fourth, by (III.),

$$d(N_{x+k+m+n}).$$

And the sum of these (divided, as usual, by D_x), is the present value required. This sum is

$$a(N_x - N_{x+k}) + b(N_{x+k} - N_{x+k+m}) + c(N_{x+k+m} - N_{x+k+m+n}) + dN_{x+k+m+n} =$$

$$aN_x + (b-a)N_{x+k} + (c-b)N_{x+k+m} + (d-c)N_{x+k+m+n} \quad . \quad \text{(XXV.)}$$

This expression shows, as will likewise readily appear from other considerations, that the benefit proposed admits of decomposition into the four following portions, viz., a life annuity of £a, and three deferred annuities of £$(b-a)$, £$(c-b)$, and £$(d-c)$, to be entered upon respectively at intervals of k, m, and n years. Also, that if b, c, and d be respectively greater than a, b, and c, the payments will increase at the expiry of each term, and *vice versâ*. If we adopt the former supposition, and suppose also $b-a=c-b=d-c=h$ (that is, that the annual rents of the deferred annuities are equal), the expression will become—

$$aN_x + h(N_{x+k} + N_{x+k+m} + N_{x+k+m+n}) \quad . \quad . \quad \text{(XXVI.)}$$

And if we further suppose $k=m=n=1$ (that is, that this uniform increase takes place at intervals of one year), it will become—

$$aN_x + h(N_{x+1} + N_{x+2} + N_{x+3});$$

which, by (6), is equal to

$$aN_x + h(S_{x+1} - S_{x+4}).$$

And this is the expression we should have derived for the benefit,

subject to these restrictions, and for the particular case in which $n=4$, from the formula [16, 12].

We have been occupied hitherto with the consideration of the present values, or single premiums, equivalent to benefits of various kinds. It is seldom, however, that benefits are paid for by a single premium. The more usual mode is by annual premiums, which may be either uniform or variable, and payable either during the whole life, or only for a specified number of years. It is necessary, therefore, now to show how the amounts of premiums, payable as described, may be found. The method of doing so constitutes the most general application of the Commutation Tables; and it is this application which most strikingly displays the great power of the method of computation we are endeavouring to illustrate, and its vast superiority to all other methods heretofore in use. It will be seen, as we proceed, that the previous portions of this paper have been but preliminary to that which forms the subject of our present remarks.

The principle upon which the application of the tables that we are now to describe depends, is so simple, that some may be disposed to award but a small portion of merit to the person by whom it was first pointed out. But as we are of those who believe that the merit of a contrivance is to be estimated in proportion to the utility of the object it has in view, and the simplicity and efficiency of the means by which that object is attained; and as the possession of those qualities, in an eminent degree, by the contrivance in question, cannot be gainsaid, we take a very different view of the matter, and presume to think the contrivance one of very great merit.

It is to Professor De Morgan that we are indebted for the contrivance we have alluded to. Others before him had, doubtless, shown how many of the problems might be solved by the Commutation Tables; but it was reserved for him to devise a general method equally applicable to all, and to exhibit an equation which should include in it almost every case, as regards both benefits and premiums, that can be proposed.*

We now proceed to explain the principle of this contrivance. The value of an annual premium whose continuance depends upon the continued existence of a given life, is evidently as legitimate a

* Mr. Davies' work having been long out of print, we have not been able to procure even a sight of it. But we have no reason to believe that it contains anything akin to the contrivance we have referred to. With the other works named on pp. 84, 85, as treating on the use of the Commutation Tables, we possess a competent acquaintance; and, certainly, in none of them is the least trace of this contrivance to be found.

subject for computation by the Commutation Tables as that of any of the benefits we have been considering. In fact, generally speaking, the only circumstances in regard to which a premium differs from an annuity for the same number of years, are, that the payments of the premium are usually made at the beginning of each year, while those of an annuity are made at the end; and that it is paid, instead of being received, by the person on whose life its continuance depends.

The last-mentioned circumstance evidently does not, in the slightest degree, affect the present value of the premium; and to the condition implied in the first we have already seen that the Commutation Tables are equally applicable as to the case of an annuity payable in the ordinary way. (See Problems II., III., and IV.)

If, therefore, we extend for a little the meaning of the term *benefit,* so as to include a payment or payments to be made by or on behalf of the party on the continuance of whose life that payment or those payments depend, then it is evident that the effecting of an assurance, or the purchase of any other benefit for which an annual premium is to be paid, is nothing more than an exchange, or *commutation* of one benefit for another depending on the same life. Consequently, that the transaction may be equitable, there must be an equality between the present values of the two benefits; or, restricting this term to its legitimate signification, between the present value of the premium and that of the benefit in exchange for which it is to be given. Therefore, if we form an equation between the expressions for the present values of the premium and the benefit, using a symbol for the unknown amount of the former, we are furnished, by the solution of this equation (which is in all cases an exceedingly simple operation), with a value of the symbol employed, which value is that of which we are in quest. It is in the idea of forming this equation that Professor De Morgan's contrivance consists, and it will be seen how efficiently it serves the intended purpose.

It now only remains for us to illustrate these observations by a few examples, after premising two or three explanatory remarks.

1. In forming the equation just referred to, it is advisable, for the sake of generalisation, to use a symbol also to denote the amount of the benefit. Those previously explained will be employed for this purpose, and π will be employed to denote the amount of a uniform premium, and the first payment of a variable premium.

2. Since a premium, as regards its present value, differs from an annuity for the same number of years only in making its first payment a year earlier, the expression for that value is found, according to the principles laid down in Problems II., III., and IV., by writing $x-1$ for x in the expression for the present value of the corresponding annuity—using, of course, π instead of a to denote the amount. Thus, the expression for the present value of a uniform life premium of £π, is πN_{x-1}; and that for a variable premium, also for life, of £$(\pi\pm\beta)$, £$(\pi\pm2\beta)$, &c., is $\pi N_{x-1}\pm\beta S_x$. Also, for the same premiums to last n years, the expressions will be

$$\pi(N_{x-1}-N_{x+n-1}),$$
$$\text{and } \pi(N_{x-1}-N_{x+n-1})\pm\beta(S_x-S_{x+n}-nN_{x+n-1}).$$

3. If we use σ to denote a sum to be paid now—that is, present value of any kind—then, to subject it to the same denominator as the other expressions for present values, viz., D_x, we must write it thus, σD_x. All the expressions for present values, as we require to use them in this application of the tables, will now be affected by the same denominator, D_x; and since, in forming an equation between any two of these expressions, this constant denominator must necessarily disappear in the solution, we have obviously no further occasion for it, and it will, therefore, never be exhibited. To avoid the anomaly, however, of longer calling the expressions from which the denominator is abstracted present values, they will be called the benefit side and the payment side of the equation.

Example 1.—Required the present value of a life assurance of £A on (x).

Here the benefit side of the equation is, by (XI.), AM_x; and the payment side is, as above, σD_x.

$$\therefore \sigma D_x=AM_x, \text{ and } \sigma=\frac{AM_x}{D_x},$$

as was before found.

Example 2.—Required π, the annual premium for a life assurance of £A on (x).

The benefit side is, as in last example, AM_x; and the payment side, by (II.), πN_{x-1}.

$$\therefore \pi N_{x-1}=AM_x, \text{ and } \pi=\frac{AM_x}{N_{x-1}}.$$

If A=£100, and $x=30$, we have

$$\pi=\frac{100M_{30}}{N_{29}}=\frac{74295\cdot67}{39382\cdot08}=1\cdot8866=\text{£}1.\ 17s.\ 9d.$$

Example 3.—An endowment of £s, to be received in n years, provided (x) be then alive, is to be paid for by an annual premium, π, to last l years. Required π.

Here the benefit side is, by (I.), sD_{x+n}; and the payment side is, by (IV.), $\pi(N_{x-1}-N_{x+l-1})$.

$$\therefore \pi(N_{x-1}-N_{x+l-1})=sD_{x+n};$$

whence,

$$\pi=\frac{sD_{x+n}}{N_{x-1}-N_{x+l-1}}.$$

If $s=£100$, $x=23$, $n=20$, and $l=10$, we have

$$\pi=\frac{100D_{43}}{N_{22}-N_{32}}=\frac{114562{\cdot}6}{58976{\cdot}15-32946{\cdot}10}=\frac{114562{\cdot}6}{26030{\cdot}05}=4{\cdot}4012=£4.\ 8s.$$

If the premium be payable till the endowment becomes due, we have only in the formula to substitute n for l; and, to apply it to the particular case just solved, we have

$$\pi=\frac{100D_{43}}{N_{22}-N_{42}}=\frac{114562{\cdot}57}{58976{\cdot}15-17483{\cdot}64}=\frac{114562{\cdot}57}{41492{\cdot}51}=2{\cdot}7610$$
$$=£2.\ 15s.\ 3d.;$$

a smaller premium, since, commencing at the same time, it continues longer.

Example 4.—£A, to be received n years hence if (x) be then alive, or at the end of the year of death if that event take place before the expiry of n years, is to be paid for by an annual premium, π, to continue until the receipt of the benefit is determined. Required π.

Here the benefit side is, by (I.) and (XIII.), $A(D_{x+n}+M_x-M_{x+n})$; and the payment side is, by (IV.), $\pi(N_{x-1}-N_{x+n-1})$.

$$\therefore \pi(N_{x-1}-N_{x+n-1})=A(D_{x+n}+M_x-M_{x+n});$$

whence,

$$\pi=\frac{A(D_{x+n}+M_x-M_{x+n})}{N_{x-1}-N_{x+n-1}}.$$

If $A=£100$, $x=30$, and $n=30$ (that is, if the benefit be receivable at 60, or at death, if before 60), we have

$$\pi=\frac{100(D_{60}+M_{30}-M_{60})}{N_{29}-N_{59}}=\frac{41857{\cdot}53+74295{\cdot}67-24598{\cdot}20}{39382{\cdot}08-4487{\cdot}42}=$$
$$\frac{91555{\cdot}00}{34894{\cdot}66}=2{\cdot}6238=£2.\ 12s.\ 6d.$$

Example 5.—Required the annual premium, π, payable till death, for an assurance of £A on (x), with which the whole of the premiums paid are to be returned.

Here the sum to be received if death take place in the first year is $A+\pi$; if in the second, $A+2\pi$; and so on. Therefore, the benefit side is, by (XIX.),

$$(A+\pi)M_x+\pi R_{x+1}=AM_x+\pi R_x, \text{ by } (10);$$

and the payment side is, by (II.), πN_{x-1}.

$$\therefore \pi N_{x-1}=AM_x+\pi R_x;$$

whence we obtain

$$\pi=\frac{AM_x}{N_{x-1}-R_x}.$$

If $A=£100$, and $x=30$, we have

$$\pi=\frac{100M_{30}}{N_{29}-R_{30}}=\frac{74295{\cdot}67}{39382{\cdot}08-17127{\cdot}42}=\frac{74295{\cdot}67}{22254{\cdot}65}=3{\cdot}3338$$
$$=£3.\ 6s.\ 8d.^*$$

Example 6.—An assurance of £A, for n years, on (x), is to be paid for by an annual premium, π, also to last n years. Required π.

Here the benefit side is, by (XIII.), $A(M_x-M_{x+n})$; and the payment side, by (IV.), $\pi(N_{x-1}-N_{x+n-1})$.

$$\therefore \pi(N_{x-1}-N_{x+n-1})=A(M_x-M_{x+n});$$

whence

$$\pi=\frac{A(M_x-M_{x+n})}{N_{x-1}-N_{x+n-1}}.$$

This expression assumes a very convenient form, if, instead of column M, we use its value in terms of N. Thus, since, by (13),

$$M_x=vN_{x-1}-N_x, \text{ and } M_{x+n}=vN_{x+n-1}-N_{x+n},$$

we have

$$\pi=\frac{A\{v(N_{x-1}-N_{x+n-1})-N_x+N_{x+n}\}}{N_{x-1}-N_{x+n-1}}=A\left(v-\frac{N_x-N_{x+n}}{N_{x-1}-N_{x+n-1}}\right).$$

* The Commutation Table, as it stands, does not enable us, conveniently, to find the amount of annual premium equivalent to a benefit which consists partly of a return of all the premiums paid, with simple interest upon them from the date of payment, the incremental portion being in this case of the form m, $3m$, $6m$, $10m$, &c. The addition of another column, formed from R as R is formed from M, would, however, afford the means of doing so. We might, obviously, add as many columns as we pleased in this way. Their properties would be such that, calling the column formed from R the *first*, the division of any number in the nth column, by the corresponding number in D, would give the present value of an assurance whose payments should be the series of figurate numbers of the nth order; and the remark may be extended, *mutatis mutandis*, to the annuity columns. But such properties being more curious than useful, we do not insist upon them. (See De Morgan, I., p. 23.)

If A=£100, $x=30$, and $n=10$, the solution by the first of these formulæ will be

$$\pi=\frac{100(M_{30}-M_{40})}{N_{29}-N_{39}}=\frac{100(742\cdot9567-522\cdot6503)}{39382\cdot08-21306\cdot62}=\frac{23030\cdot64}{18075\cdot46}=$$

$$1\cdot2188=£1.\ 4s.\ 5d.$$

The solution by the second formula will be

$$\pi=100\left(v-\frac{N_{30}-N_{40}}{N_{29}-N_{39}}\right)=100\left(\cdot961539-\frac{17159\cdot95}{18075\cdot46}\right)=$$

$$100(\cdot961539-\cdot949351)=1\cdot2188,$$

as before.

Example 7.—An annuity of £a, deferred for n years, on (x), is to be paid for by a uniformly decreasing annual premium, to be extinguished when the annuity is entered upon. Required π, the first year's premium.

Here the benefit side is, by (III.), aN_{x+n}; and, since the premium is to be extinguished in n years, it will make $n+1$ payments. In order, therefore, that the $(n+2)$th payment, which would be due when the first payment of the annuity is receivable, may be 0, the annual decrease must be $\frac{\pi}{n+1}$. The payment side consequently is, by [20, 6],

$$\pi\{N_{x-1}-\frac{1}{n+1}(S_x-S_{x+n+1})\}.$$

Hence,

$$\pi\{N_{x-1}-\frac{1}{n+1}(S_x-S_{x+n+1})\}=aN_{x+n}.$$

$$\therefore\ \pi=\frac{aN_{x+n}}{N_{x-1}-\frac{1}{n+1}(S_x-S_{x+n+1})}.$$

If $a=100$, $x=40$, and $n=20$, we have

$$\pi=\frac{100N_{60}}{N_{39}-(S_{40}-S_{61})\div21}=\frac{406884\cdot83}{21306\cdot62-(252533\cdot78-29648\cdot60)\div21}$$

$$=\frac{406884\cdot83}{21306\cdot62-10613\cdot06}=\frac{406884\cdot83}{10693\cdot56}=38\cdot0495=£38.\ 1s.$$

Example 8.—An increasing life assurance of £A, £(A+H), £(A+2H), &c., on (x), is to be paid for by a premium which is to

be π for the first n years, $\frac{2}{3}\pi$ for the next n years, and $\frac{1}{3}\pi$ for the remainder of life. Required π.

The benefit side is, by (XIX.), AM_x+HR_{x+1}; and the payment side is—

For the first n years,	$\pi(N_{x-1}-N_{x+n-1})$;
,, second,	$\frac{2}{3}\pi(N_{x+n-1}-N_{x+2n-1})$;
Remainder of life,	$\frac{1}{3}\pi N_{x+2n-1}$.

Therefore, adding these for the whole payment side, we have

$$\pi\{N_{x-1}-\tfrac{1}{3}(N_{x+n-1}+N_{x+2n-1})\}=AM_x+HR_{x+1}.$$

$$\therefore \pi=\frac{AM_x+HR_{x+1}}{N_{x-1}-\frac{1}{3}(N_{x+n-1}+N_{x+2n-1})}.$$

If $A=£100$, $H=£5$, $x=30$, and $n=7$, the formula becomes

$$\pi=\frac{100M_{30}+5R_{31}}{N_{29}-\frac{1}{3}(N_{36}+N_{43})}.$$

And, taking the numbers from the table, we should find

$$\pi=6{\cdot}1644=£6.\ 3s.\ 3d.$$

This mode of paying premium, although not unusual in practice, is not included in Professor De Morgan's general problem (I., pp. 10, 11). We subjoin the general expression for the payment side of the equation when a benefit is to be paid for in this manner—that is, by a premium remaining constant during one or more terms of years, but varying, either by increase or decrease, at the end of each term. Let p, q, and r, denote any numbers whatsoever, whether whole or fractional; also, let l, m, and n, denote any terms of years whatsoever; then the payment side of the equation, for a premium which is to be π during the first l years, $p\pi$ during the following m years, $q\pi$ during the following n years, and $r\pi$ for the remainder of life, will be

$$\pi\{N_{x-1}+(p-1)N_{x+l-1}+(q-p)N_{x+l+m-1}+(r-q)N_{x+l+m+n-1}\}.$$

From this expression it appears, that, if p be less than unity, q less than p, and r less than q, the premium will decrease at the end of each term, and *vice versâ*. To apply this expression to the last example, let $x=30$, $p=\frac{2}{3}$, $r=\frac{1}{3}$, $l=m=7$; and, since there

are only two *definite* periods, during which the payments are uniform, $n=0$ and $q=0$. Hence the expression becomes

$$\pi(N_{29}-\tfrac{1}{3}N_{36}-\tfrac{2}{3}N_{43}+\tfrac{1}{3}N_{43})=\pi\{N_{29}-\tfrac{1}{3}(N_{36}+N_{43})\}.$$

Example 9.—A life assurance of £A on (x) is to be paid for by a sum in hand, σ, and an annual premium, π. Required π.

Here the benefit side is AM_x, and the payment side is $\sigma D_x+\pi N_{x-1}$.

$$\therefore \sigma D_x+\pi N_{x-1}=AM_x;$$

whence,

$$\pi=\frac{AM_x-\sigma D_x}{N_{x-1}}.$$

If $A=$£1,000, $x=40$, and $\sigma=$£100, we should find

$$\pi=18\cdot2308=£18.\ 4s.\ 7d.$$

Example 10.—A person now aged x years effected an assurance on his life of £A, n years ago, at an annual premium of £π, of which a payment is just due. He now wishes to dispose of his interest in the same, the purchaser to take on himself the payment of the future premiums, and to receive the sum assured when it becomes due by the death of (x). Required σ, the sum to be paid to (x) for his interest in the policy.

The sum to be paid to (x)—that is, the present value of the policy—is evidently the difference between the present value of the assurance and that of the future premiums.

$$\therefore \sigma D_x=AM_x-\pi N_{x-1};$$

whence,

$$\sigma=\frac{AM_x-\pi N_{x-1}}{D_x}.$$

If $A=$£100, and $x=40$, then, if the assurance was effected ten years ago, and the premium calculated by the same table as is made use of for the valuation of the policy,* we should have

$$\pi=\frac{AM_{30}}{N_{29}}=1\cdot8864.$$

Hence we find

$$\sigma=8\cdot9926=£8.\ 19s.\ 10d.$$

Here we must stop. The object of the writer, in commencing the present paper, was, as stated in the outset, to furnish an easy

* This is by no means necessarily the case. The *actual* premium must always be used.

introduction to the papers on the same subject by Professor De Morgan, in the *Companion to the Almanack;* as he had found, in the course of his experience, that persons even who had paid some attention to the subject were at a loss to comprehend the scope of the papers alluded to. Whether or not he has in any degree succeeded in his object, is for those to say who may have honoured his lucubrations with an attentive perusal. He is himself sensible of many deficiencies in them, some of which, perhaps, if the opportunity were afforded him, he might be able to amend. Such as they are, however, they are now before the public; and he trusts that, as the work of an amateur, they will be viewed with indulgence. In particular, he has to bespeak the forbearance of the learned individual, in the desire to render whose writings on the subject more accessible than heretofore, they have originated. It is only now, in taking leave of the subject, that he perceives, in its full force, the presumption of which he fears he has been guilty. But as he thinks it may be allowed that his object has been praiseworthy, he trusts this will be received as an atonement for deficiencies in other respects. In the preparation of the article the writer has made free use of Professor De Morgan's papers, as indeed he could hardly fail to do, since to them and to Mr. Jones's work he owes all the information he possesses on the subject. Notwithstanding this, however, and the circumstance that his paper considerably exceeds in length those of the learned Professor, the vein from which the materials have been taken is very far indeed from being exhausted. It will amply repay a search on the part of those whom interest or inclination may lead to cultivate this branch of science.

Erratum.—Page 104, line 2 from top, *dele* "or n into $k+n$." The writer is not aware of any other error affecting any of the formulæ, nor has he found an error in the table as printed.

NOTES AND QUERIES.

Mr. Robert Christie, in a letter to Mr. Samuel Brown, says:—"It has occurred to me that it may contribute to the object in view" (that is, to the collection of materials for framing a table of mortality based on the experience of the Offices) "if the extent of life assurance actually transacted in the United Kingdom were ascertained.

"With this view, having applied to each of the native Scottish Life

Offices* for information, I am now enabled to state the present aggregate of their annual income, their realised fund, and the amount assured. These particulars, carefully excluding fire insurance and all income derived from it, are the following:—

Annual income from premiums, and from realised or accumulated funds	£2,200,933
Realised or accumulated funds	12,807,057
Amount presently assured, including bonuses	54,692,877

"I hope you may be able to obtain similar information applicable to the London and other Offices."

The Public Debt due to the Bank.

			£	s.	d.
1694	Original subscription lent to Government at 8 per cent. per annum		1,200,000	0	0
1708	Advanced to Government without interest		400,000	0	0
„	Exchequer Bills cancelled		1,775,027	17	10
1717	Ditto ditto		2,000,000	0	0
1722	Advanced to pay off South Sea stockholders		4,000,000	0	0
			9,375,027	17	10
1727-8	Deduct amount paid from the Sinking Fund		1,775,027	17	10
			7,600,000	0	0
1728	Advanced on the security of taxes		1,750,000	0	0
„	Ditto upon lottery		1,250,000	0	0
			10,600,000	0	0
„	Paid from Sinking Fund	£500,000			
1738	Ditto ditto	1,000,000			
			1,500,000	0	0
			9,100,000	0	0
1742	Advanced without interest		1,600,000	0	0
1746	Exchequer Bills cancelled		986,800	0	0
1816	Advanced at 3 per cent. interest		3,000,000	0	0
			14,686,800	0	0

* Scottish Life Offices above referred to.

Scottish Widows' Fund and Life Assurance Society	Instituted	1815
Edinburgh Life Company	„	1823
North British (Life Branch)	„	1823
Scottish Union	„	1824
Standard	„	1825
Scottish Provincial	„	1825
Scottish Amicable	„	1826
Scottish Equitable	Instituted	1831
Caledonian (Life Branch)	„	1833
Northern	„	1836
Scottish Provident	„	1837
Life Association of Scotland	„	1838
City of Glasgow	„	1838
Scottish National	„	1841
Colonial	„	1846

		£	s.	d.
	Brought forward . . .	14,686,800	0	0
1835	By the transfer of £4,080,000 Reduced Three per Cent. Annuities, equal to one-fourth of the debt, in accordance with the 9th section of the Charter Act of 1833 (3 & 4 Will. IV., cap. 98)	3,671,700	0	0
	Total debt . . .	£11,015,100	0	0

The sum of £2,984,900 has since been added to make up the £14,000,000; and the remaining £650,000 have been added under the Act of 1844, which empowers the Bank to issue equal to *two-thirds* of the country issues withdrawn from circulation.—*Money Market Review.*

Statistics of Human Life.—The total number of human beings on earth is now computed in round numbers at 1,000,000,000. They speak 3,064 now known tongues, and in which upwards of 1,100 religions or creeds are preached. The average age of life is $33\frac{1}{3}$ years. One-fourth of the born die before they reach the age of 7 years, and the half before the 17th year. Out of 100 persons only 6 reach the age of 60 years and upwards, while only 1 in 1,000 reaches the age of 100 years. Out of 500 only 1 attains 80 years. Out of the thousand million living persons 330,000,000 die annually, 91,000 daily, 3,730 every hour, 60 every minute, consequently 1 every second. The loss is, however balanced by the gain in new births. Tall men are supposed to live longer than short ones. Women are generally stronger than men until their 50th year, afterwards less so. Marriages are in proportion to single life (bachelors and spinsters) as 100 : 75. Both births and deaths are more frequent in the night than in the day. One-fourth of men are capable of bearing arms, but not 1 out of 1,000 is by nature inclined for the profession. The more civilised a country is, the more full of vigour, life, and health are the people. The notion that education enfeebles and degenerates the human frame is not borne out by fact.—*Weekly Paper.*

CORRESPONDENCE.

MR. WOOLHOUSE'S RECENT PAPER.

To the Editor of the Assurance Magazine.

My dear Sir,—Mr. Woolhouse is quite accurate in his investigation (*ante*, p. 128), but whether he accurately interprets my meaning of 1839, is more than I can now tell. When I say that it is not necessary that the progression should be "precisely" that of Mr. Gompertz, there seems to be an indisposition to affirm that it can very widely depart. When I then proceed to give a very general equation, there seems to be an implication that there is something like an extensive form of solution. I suppose that

I intended to announce the general equation, without taking the trouble to investigate any possible limitations, but without denying that limitations may exist. This is the conclusion about my own meaning to which I came three years ago, when I inverted the question (vol. viii., p. 181) and reduced it to the solution of a functional equation. Mr. Woolhouse cannot have been aware* of this second paper of mine, in which is "required the law of mortality under which the table of two lives follows the same law as the table of one life." The demonstrated conclusion is that nothing but Gompertz's law will do. If anyone should happen to know—as was the fact—that I examined the proof of your reprint of my old paper, he will think that I ought to have made an allusion to the recent paper of a more direct character than yours. I am very glad I did not; the consequence has been that we have Mr. Woolhouse's simple algebraical treatment of the subject, which is quite within the grasp of an elementary student.

One thing, however, is wanted: Mr. Woolhouse's solution of the functional equation is good for integer differences of n, but the law may vary in an infinite number of ways during the parts of a year. All that is needed is to notice that the *unit of time* may be any whatsoever. The same caution applies to many cases in which functional equations are employed.

In this last remark I am supposing that Mr. Woolhouse, as is usual, contemplates such a *curve* of mortality as can be laid down by the help of the numbers living at the end of each year, and by the usual principles of interpolation. These principles contain the assumption that there are no inequalities whose cycle is precisely equal to the interval of time by which values are separated. When the moon's right ascension, obtained from theory for a succession of noons and midnights, is thence obtained by interpolation at all the intermediate hours, it is assumed that there are no perceptible inequalities which have a cycle of twelve hours of mean time; and the assumption being true, the method answers. It is customary to overlook any inequalities of mortality which run their course within a year, but this neglect will not endure for ever. The changes of season have a much more sensible effect on health and life than the rotation of the earth has on the moon's motion. The time may come when the trigonometrical considerations which enter into the *complete* solution of the functional equation may be called into use; and the number living at the age x may be a function of the sun's right ascension, and perhaps of the moon's right ascension also. In the mean time the limited solution of the functional equation, when the true grounds of its practical sufficiency are pointed out, may be made a useful lesson to those who are beginning the subject.

I am, yours very truly,

April 24th, 1862. A. DE MORGAN.

ANOTHER DEMONSTRATION OF THE EXPRESSIONS FOR THE VALUE OF SINGLE AND ANNUAL PREMIUMS.

To the Editor of the Assurance Magazine.

Sir,—Can you make room for yet another determination of the single and annual premiums for assurance?

* Mr. Woolhouse was not aware of it when he wrote.—Ed. *A. M.*

An assurance of £1 is to be effected on (x), for which the single premium is A_x. This premium is not to be paid now, however, but is to remain a debt upon the policy till the latter becomes a claim; and an equitable consideration is to be paid annually, during the life of (x), for its forbearance. Now, the consideration for forbearance of £1 for a year being $1-v$, that for forbearance of A_x for the same time will be $(1-v)A_x$; and this, therefore, will be the annual payment to be made during the life of (x).

Hence the total consideration to be made for the assurance of £1 will be, first, a payment at the end of the year of death of A_x, the sum forborne, the present value of which is $A_x \times A_x$, or A_x^2; and, second, an annual payment of $(1-v)A_x$, the present value of which is $(1-v)a'_x A_x$, where a'_x denotes the present value of an annuity due of £1 on (x).

Therefore, equating present values,

$$A_x = A_x^2 + (1-v)a'_x A_x;$$

whence, dividing by A_x and transposing,

$$A_x = 1-(1-v)a'_x.$$

Since the claims arising by the death of (x) are satisfied by a payment from the Office of $1-A_x$, it follows that the foregoing transaction resolves itself into an assurance of $1-A_x$, paid for by the annual premium $(1-v)A_x$. Hence, if π_x denote the annual premium for an assurance of £1, we have

$$1-A_x : 1 :: (1-v)A_x : \pi_x$$

$$\therefore \pi_x = \frac{(1-v)A_x}{1-A_x},$$

a known form, which reduces to the more usual form by substitution for A_x in terms of a'_x.

I am, Sir,

Your most obedient servant,

P. GRAY.

7, *St. Paul's Villas, Camden Town,*
4th June, 1862.

INSTITUTE OF ACTUARIES.

PROCEEDINGS OF THE INSTITUTE.

Third Ordinary Meeting, Session 1861-62.—*Monday, 27th January,* 1862.

The President in the Chair.

The minutes of the last ordinary meeting were read and confirmed.

The Secretary announced various donations to the library.

The undermentioned gentlemen, duly nominated at the last ordinary meeting, were elected members of the Institute :—

Fellow—Robert Thomson, Esq.

Official Associate—J. L. Pilkington, Esq., F.S.S.

Associates.

Mr. M. P. Christie, B.A.	Mr. F. J. C. Taylor.
„ E. Justican.	„ Fred. Terry.
„ W. E. S. Macdonald.	„ Chas. Woolhouse.

Mr. James Terry read a paper "On the tendency of some systems of distribution of surplus to defeat the object of life assurance."

Thanks were voted to Mr. Terry, and the meeting adjourned to 24th February, 1862.

Fourth Ordinary Meeting, Session 1861-62.—*Monday,* 24*th February,* 1862.

The President in the Chair.

The minutes of the last ordinary meeting were read and confirmed.

The Secretary announced various donations to the library.

Mr. Charles Weeks, duly nominated at the last ordinary meeting, was unanimously elected an Associate of the Institute.

Mr. Bailey read a paper "On the principles on which the funds of Life Assurance Societies should be invested."

Thanks were voted to Mr. Bailey, and the meeting adjourned to Monday, 24th March, 1862.

Fifth Ordinary Meeting, Session 1861-62.—*Monday,* 24*th March,* 1862.

The President in the Chair.

The minutes of the last ordinary meeting were read and confirmed.

The Secretary announced various donations to the library.

Mr. William D. Biden, duly nominated at the last ordinary meeting, was unanimously elected an Associate of the Institute.

Mr. Archibald Day read a paper "On the statistics of first and subsequent marriages among the families of the peerage, considered specially with reference to the calculation of premiums for assurances against issue."

Thanks having been voted to Mr. Day, the meeting adjourned to Monday, 28th April, 1862.

Sixth Ordinary Meeting, Session 1861-62.—*Monday,* 28*th April,* 1862.

The President in the Chair.

The minutes of the last ordinary meeting were read and confirmed.

The Secretary announced various donations to the library.

Mr. William B. Row, B.A., duly nominated at the last ordinary meeting, was unanimously elected an Official Associate of the Institute.

Mr. Newmarch delivered an address "On the probable future of the rate of interest in this country."

Thanks having been voted to Mr. Newmarch, the meeting adjourned to Monday, 24th November, 1862.

THE

ASSURANCE MAGAZINE,

AND

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

Solutions of the Compound Survivorship Assurance Problems. By WILLIAM MATTHEW MAKEHAM, *of the Church of England Assurance Institution.*

THE problems referred to under the above designation are those which involve the double survivorship contingency of *A* living after the death of *B* before *C*. The following solutions of these cases are quite general, and consequently applicable to any mortality table whatever.

I have taken the problems in the order in which they appear in the work of Mr. Milne, whose notation I have also followed. By means of an obvious relation which exists between the value of the compound survivorship assurance $\underset{\mathrm{III}}{\mathfrak{A}}\underset{\mathrm{II}}{\mathbb{B}}\mathbb{C}$ and the corresponding annuity, the whole of these cases are reduced to the single fundamental annuity value $_{\rceil \underset{1}{\mathrm{B}}\mathrm{C}}\mathrm{A}$, combined with the values of simple survivorship assurances. Consequently, if we should ever possess a complete table of the values of the simple survivorship assurance $\underset{1}{\mathfrak{A}}\mathbb{B}\mathbb{C}$, and also of the values of the annuity $_{\rceil \underset{1}{\mathrm{B}}\mathrm{C}}\mathrm{A}$, we shall be able to calculate the numerical value in any case with the greatest facility.

Although it must be confessed that there exists, at present, but

little probability of such a task being undertaken as the construction of the tables above referred to, it may be observed that the quantity of the tabular matter required in the second case is considerably reduced in consequence of the equation $_{\rceil\underset{1}{BC}}A = A - ABC - {}_{\rceil\underset{1}{BC}}A$.

The following "Example" from the work of Mr. Griffith Davies is a good illustration of the class of cases which a table of the values of the fundamental annuity $_{\rceil\underset{1}{BC}}A$ is calculated to meet:—

"Suppose *B*, aged 35, to be entitled to an estate in the event of his surviving his father, *C*, aged 60, and desirous of providing an annuity of £500 to his wife *A*, aged 30, to commence at his death in the event of his dying before his father; it is required to determine the present value of *A*'s contingent annuity." And Mr. Davies finds that the actual value of such an annuity is less than two years' purchase.

When it is considered how frequently cases of the above description must occur, and at how little cost a provision can be secured against the contingency to which the wife is exposed, it must be a matter of surprise that assurances of the kind referred to in the above "Example" are not more frequently effected. The explanation is probably to be found in the fact that the general public are but little aware with what precision the science of life contingencies, when the requisite tables have been computed, is capable of meeting the various complications in which contingent life interests are frequently involved.

I have further to observe that the formulæ obtained by the following method of solution may also be deduced from the functional expressions to which Mr. Gompertz reduces these cases in his *Analysis applicable to the estimation of Life Contingencies.*

Problem I. (*Milne*, 21; *Baily*, 42).

To determine $(\underset{\text{III}}{\mathfrak{A}}\underset{\text{II}}{\mathbb{B}}\mathbb{C})$ the present value of £1 payable upon *A* failing last of the three lives *A, B, C,* provided that *B* fail second.

Solution.

It is evident that an annuity which is to commence at the death of *B*, in the event of his dying before *A* and after *C*, and then to continue during the remainder of *A*'s life, together with a perpetuity which is to commence at the failure of that annuity (*i.e.*, at *A*'s death under the contingency named in the problem), will be precisely the same as a perpetuity which is to commence at

the death of B in the event of his dying before A and after C. That is—

$${}_{\rceil\overline{\underset{\text{II}}{B}C}}A+\underset{\text{III}}{\mathfrak{A}}\underset{\text{II}}{\mathbb{B}}\mathbb{C}(1+\tfrac{1}{r})=\underset{\text{II}}{\mathfrak{B}}\mathbb{A}\underset{\text{I}}{\mathbb{C}}(1+\tfrac{1}{r});\text{ and, therefore,}$$

$$\underset{\text{III}}{\mathfrak{A}}\underset{\text{II}}{\mathbb{B}}\mathbb{C}=\underset{\text{II}}{\mathfrak{B}}\mathbb{A}\underset{\text{I}}{\mathbb{C}}-{}_{\rceil\overline{\underset{\text{II}}{B}C}}A(1-v).$$

But $\underset{\text{II}}{\mathfrak{B}}\mathbb{A}\underset{\text{I}}{\mathbb{C}}=\underset{\text{I}}{\mathfrak{B}}\mathbb{A}-\underset{\text{I}}{\mathfrak{B}}\mathbb{A}\mathbb{C}$ (Milne, Prob. 19). And

$${}_{\rceil\overline{\underset{\text{II}}{B}C}}A(1-v)=A(1-v)-AB(1-v)-{}_{\rceil\underset{\text{I}}{B}C}A(1-v)\text{ (Milne, Prob. 6)}$$

$$=\mathfrak{A}\mathfrak{B}-\mathfrak{A}-{}_{\rceil\underset{\text{I}}{B}C}A(1-v).$$

Therefore, $\underset{\text{III}}{\mathfrak{A}}\underset{\text{II}}{\mathbb{B}}\mathbb{C}=\underset{\text{I}}{\mathfrak{B}}\mathbb{A}-\underset{\text{I}}{\mathfrak{B}}\mathbb{A}\mathbb{C}-\mathfrak{A}\mathfrak{B}+\mathfrak{A}+{}_{\rceil\underset{\text{I}}{B}C}A(1-v)$

$$=\underset{\text{II}}{\mathfrak{A}}\mathbb{B}-\underset{\text{I}}{\mathfrak{B}}\mathbb{A}\mathbb{C}+{}_{\rceil\underset{\text{I}}{B}C}A(1-v)\quad .\quad .\quad .\quad [1]$$

This expression may be put in another form, thus :—

$${}_{\rceil\underset{\text{I}}{B}C}A+{}_{\rceil B\underset{\text{I}}{C}}A={}_{\rceil BC}A=A-ABC$$

$$\therefore\ {}_{\rceil\underset{\text{I}}{B}C}A(1-v)=A(1-v)-ABC(1-v)-{}_{\rceil B\underset{\text{I}}{C}}A(1-v)$$

$$=\mathfrak{A}\mathfrak{B}\mathfrak{C}-\mathfrak{A}-{}_{\rceil B\underset{\text{I}}{C}}A(1-v).$$

Substituting in [1], we have

$$\underset{\text{III}}{\mathfrak{A}}\underset{\text{II}}{\mathbb{B}}\mathbb{C}=\mathfrak{A}\mathfrak{B}\mathfrak{C}-\underset{\text{I}}{\mathfrak{B}}\mathbb{A}\mathbb{C}-\underset{\text{I}}{\mathfrak{A}}\mathbb{B}-{}_{\rceil B\underset{\text{I}}{C}}A(1-v)$$

$$=\underset{\text{I}}{\mathfrak{A}}.\underset{\text{I}}{\mathfrak{C}}\mathbb{B}-\underset{\text{I}}{\mathfrak{A}}\mathbb{B}-{}_{\rceil B\underset{\text{I}}{C}}A(1-v)\quad .\quad .\quad .\quad [2]$$

PROBLEM II. (*Milne*, 22; *Baily*, 45).

To determine $\left(\underset{\substack{\text{I.III}\\ cb\\ \text{I}}}{\mathfrak{A}}\mathbb{B}\mathbb{C}\right)$ the present value of £1 payable upon A failing either first or last of the three lives A, B, C; provided that, in the latter case, C fail first.

Solution.

$$\underset{\substack{\text{I.III}\\ cb\\ \text{I}}}{\mathfrak{A}}\mathbb{B}\mathbb{C}=\underset{\text{I}}{\mathfrak{A}}\mathbb{B}\mathbb{C}+\underset{\text{III}}{\mathfrak{A}}\underset{\text{II}}{\mathbb{B}}\mathbb{C}.$$

Substituting the first value of $\underset{\text{III}}{\mathfrak{A}}\underset{\text{II}}{\mathbb{B}}\mathbb{C}$, we have

$$\underset{\substack{\text{I.III}\\ cb\\ \text{I}}}{\mathfrak{A}}\mathbb{B}\mathbb{C}=\underset{\text{I}}{\mathfrak{A}}\mathbb{B}\mathbb{C}+\underset{\text{II}}{\mathfrak{A}}\mathbb{B}-\underset{\text{I}}{\mathfrak{B}}\mathbb{A}\mathbb{C}+{}_{\rceil\underset{\text{I}}{B}C}A(1-v)\quad .\quad .\quad [1]$$

Again, substituting in the first equation the second value of $\underset{\text{III}}{\mathfrak{A}}\underset{\text{II}}{\mathbb{B}}\mathbb{C}$, we have

$$\underset{\substack{\text{I.III}\\ cb\\ \text{I}}}{\mathfrak{A}}\mathbb{B}\mathbb{C}=\underset{\text{I}}{\mathfrak{A}}\mathbb{B}\mathbb{C}+\underset{\text{I}}{\mathfrak{A}}.\underset{\text{I}}{\mathfrak{C}}\mathbb{B}-\underset{\text{I}}{\mathfrak{A}}\mathbb{B}-{}_{\rceil B\underset{\text{I}}{C}}A(1-v)$$

$$=2\underset{\text{I}}{\mathfrak{A}}\mathbb{B}\mathbb{C}+\underset{\text{I}}{\mathfrak{C}}\mathbb{A}\mathbb{B}-\underset{\text{I}}{\mathfrak{A}}\mathbb{B}-{}_{\rceil B\underset{\text{I}}{C}}A(1-v)\quad .\quad .\quad [2]$$

PROBLEM III. (*Milne*, 23; *Baily*, 44).

To determine $(\underset{\text{ii.iii}}{\mathfrak{A}}\mathbb{B}\underset{\text{i}}{\mathbb{C}})$ the present value of £1 payable upon *A* failing either second or third, provided that *C* fail first of the three lives *A*, *B*, *C*.

Solution.

$$\begin{aligned}\underset{\text{ii.iii}}{\mathfrak{A}}\mathbb{B}\underset{\text{i}}{\mathbb{C}}&=\underset{\text{ii}}{\mathfrak{A}}\mathbb{B}\underset{\text{i}}{\mathbb{C}}+\underset{\text{iii}}{\mathfrak{A}}\underset{\text{ii}}{\mathbb{B}}\mathbb{C}\\&=\underset{\text{i}}{\mathfrak{A}}\mathbb{B}-\underset{\text{i}}{\mathfrak{A}}\mathbb{B}\mathbb{C}+\underset{\text{iii}}{\mathfrak{A}}\underset{\text{ii}}{\mathbb{B}}\mathbb{C}\end{aligned}$$

Substituting the first value of $\underset{\text{iii}}{\mathfrak{A}}\underset{\text{ii}}{\mathbb{B}}\mathbb{C}$, we get

$$\begin{aligned}\underset{\text{ii.iii}}{\mathfrak{A}}\mathbb{B}\underset{\text{i}}{\mathbb{C}}&=\underset{\text{i}}{\mathfrak{A}}\mathbb{B}-\underset{\text{i}}{\mathfrak{A}}\mathbb{B}\mathbb{C}+\underset{\text{ii}}{\mathfrak{A}}\mathbb{B}-\underset{\text{i}}{\mathfrak{B}}\mathbb{A}\mathbb{C}+{}_{\rceil\mathrm{B}\underset{\text{i}}{\mathrm{C}}}\mathrm{A}(1-v)\\&=\mathfrak{A}-\underset{\text{i}}{\mathfrak{A}}.\underset{\text{i}}{\mathfrak{B}}\mathbb{C}+{}_{\rceil\mathrm{B}\underset{\text{i}}{\mathrm{C}}}\mathrm{A}(1-v)\quad . \quad . \quad . \quad . \quad [1]\end{aligned}$$

Again, substituting the second value of $\underset{\text{iii}}{\mathfrak{A}}\underset{\text{ii}}{\mathbb{B}}\mathbb{C}$, we have

$$\begin{aligned}\underset{\text{ii.iii}}{\mathfrak{A}}\mathbb{B}\underset{\text{i}}{\mathbb{C}}&=\underset{\text{i}}{\mathfrak{A}}.\underset{\text{i}}{\mathfrak{C}}\mathbb{B}-\underset{\text{i}}{\mathfrak{A}}\mathbb{B}\mathbb{C}-{}_{\rceil\mathrm{B}\underset{\text{i}}{\mathrm{C}}}\mathrm{A}(1-v)\\&=\underset{\text{i}}{\mathfrak{C}}\mathbb{A}\mathbb{B}-{}_{\rceil\mathrm{B}\underset{\text{i}}{\mathrm{C}}}\mathrm{A}(1-v)\quad . \quad . \quad . \quad . \quad [2]\end{aligned}$$

PROBLEM IV. (*Milne*, 24; *Baily*, 47).

To determine $(\overline{\mathfrak{A}\mathfrak{B}}\underset{\text{i}}{\mathbb{C}})$ the present value of £1 payable upon the failure of the last survivor of the two lives *A*, *B*, provided they both survive a third life, *C*.

Solution.

$$\overline{\mathfrak{A}\mathfrak{B}}\underset{\text{i}}{\mathbb{C}}=\underset{\text{iii}}{\mathfrak{A}}\underset{\text{ii}}{\mathbb{B}}\mathbb{C}+\underset{\text{iii}}{\mathfrak{B}}\underset{\text{ii}}{\mathbb{A}}\mathbb{C}.$$

Substituting the first value of $\underset{\text{iii}}{\mathfrak{A}}\underset{\text{ii}}{\mathbb{B}}\mathbb{C}$ and the second value of $\underset{\text{iii}}{\mathfrak{B}}\underset{\text{ii}}{\mathbb{A}}\mathbb{C}$, we have (putting for $\underset{\text{ii}}{\mathfrak{A}}\mathbb{B}$ its equivalent $\mathfrak{A}-\underset{\text{i}}{\mathfrak{A}}\mathbb{B}$),

$$\overline{\mathfrak{A}\mathfrak{B}}\underset{\text{i}}{\mathbb{C}}=\mathfrak{A}-\underset{\text{i}}{\mathfrak{A}}\mathbb{B}-\underset{\text{i}}{\mathfrak{B}}\mathbb{A}\mathbb{C}+{}_{\rceil\mathrm{B}\underset{\text{i}}{\mathrm{C}}}\mathrm{A}(1-v)+\underset{\text{i}}{\mathfrak{B}}.\underset{\text{i}}{\mathfrak{C}}\mathbb{A}-\underset{\text{i}}{\mathfrak{B}}\mathbb{A}-{}_{\rceil\mathrm{A}\underset{\text{i}}{\mathrm{C}}}\mathrm{B}(1-v);$$

and, finally,

$$\overline{\mathfrak{A}\mathfrak{B}}\underset{\text{i}}{\mathbb{C}}=\underset{\text{i}}{\mathfrak{C}}\mathbb{A}\mathbb{B}-\mathfrak{A}\mathfrak{B}+\mathfrak{A}+({}_{\rceil\mathrm{B}\underset{\text{i}}{\mathrm{C}}}\mathrm{A}-{}_{\rceil\mathrm{A}\underset{\text{i}}{\mathrm{C}}}\mathrm{B})(1-v).$$

PROBLEM V. (*Milne*, 26; *Baily*, 48).

To determine $(\overline{\mathfrak{A}\mathfrak{B}}\underset{\text{ii}}{\mathbb{C}})$ the present value of £1 payable upon the failure of the last survivor of the two lives *A*, *B*, provided that one of them fail before, and the other after, a third life, *C*.

Solution.

$$\text{Since }\overline{\mathfrak{A}\mathfrak{B}}\underset{\text{i}}{\mathbb{C}}+\overline{\mathfrak{A}\mathfrak{B}}\underset{\text{ii}}{\mathbb{C}}+\overline{\mathfrak{A}\mathfrak{B}}\underset{\text{iii}}{\mathbb{C}}=\overline{\mathfrak{A}\mathfrak{B}}$$

$$\overline{\mathfrak{A}\mathfrak{B}}\underset{\text{ii}}{\mathbb{C}}=\overline{\mathfrak{A}\mathfrak{B}}-\overline{\mathfrak{A}\mathfrak{B}}\underset{\text{i}}{\mathbb{C}}-\overline{\mathfrak{A}\mathfrak{B}}\underset{\text{iii}}{\mathbb{C}}$$

$$\text{But }\overline{\mathfrak{A}\mathfrak{B}}\underset{\text{iii}}{\mathbb{C}}=\underset{\text{i}}{\mathfrak{A}}\mathbb{C}+\underset{\text{i}}{\mathfrak{B}}\mathbb{C}-\underset{\text{i}}{\mathfrak{A}}.\underset{\text{i}}{\mathfrak{B}}\mathbb{C}\ \text{(Milne, Prob. 25).}$$

Substituting this value, and that of $\overline{\mathfrak{AB}}\underset{\text{i}}{\mathbb{C}}$, as found in the last problem, we have

$$\overline{\mathfrak{AB}}\underset{\text{ii}}{\mathbb{C}}=\overline{\mathfrak{AB}}-\mathfrak{A}+\mathfrak{AB}-\underset{\text{i}}{\mathfrak{C}}\mathbb{AB}-({}_{\rceil\underset{1}{BC}}A-{}_{\rceil\underset{1}{AC}}B)(1-v)-\underset{\text{i}}{\mathfrak{A}}\mathbb{C}-\underset{\text{i}}{\mathfrak{B}}\mathbb{C}+\underset{\text{i}}{\mathfrak{A}}.\underset{\text{i}}{\mathfrak{B}}\mathbb{C}$$

$$=\mathfrak{B}-\underset{\text{i}}{\mathfrak{A}}\mathbb{C}-\underset{\text{i}}{\mathfrak{B}}\mathbb{C}+\mathfrak{ABC}-2\underset{\text{i}}{\mathfrak{C}}\mathbb{AB}-({}_{\rceil\underset{1}{BC}}A-{}_{\rceil\underset{1}{AC}}B)(1-v).$$

Again, substituting for ${}_{\rceil\underset{1}{BC}}A(1-v)$ its equivalent $\mathfrak{ABC}-\mathfrak{A}-{}_{\rceil\underset{1}{BC}}A(1-v)$, we get

$$\overline{\mathfrak{AB}}\underset{\text{ii}}{\mathbb{C}}=\mathfrak{A}+\mathfrak{B}-\underset{\text{i}}{\mathfrak{A}}\mathbb{C}-\underset{\text{i}}{\mathfrak{B}}\mathbb{C}-2\underset{\text{i}}{\mathfrak{C}}\mathbb{AB}+({}_{\rceil\underset{1}{BC}}A+{}_{\rceil\underset{1}{AC}}B)(1-v).$$

Problem VI. (*Milne*, 28).

To determine $(\overline{\mathfrak{AB}}\underset{\text{i.iii}}{\mathbb{C}})$ the present value of £1 payable upon the failure of the survivor of the two lives *A, B*, provided that a third life, *C*, fail either first or last of the three.

Solution.

$$\overline{\mathfrak{AB}}\underset{\text{i.iii}}{\mathbb{C}}=\overline{\mathfrak{AB}}-\overline{\mathfrak{AB}}\underset{\text{ii}}{\mathbb{C}}.$$

Substituting for $\overline{\mathfrak{AB}}\underset{\text{ii}}{\mathbb{C}}$ its value as found in the last problem, we have

$$\overline{\mathfrak{AB}}\underset{\text{i.iii}}{\mathbb{C}}=\underset{\text{i}}{\mathfrak{A}}\mathbb{C}+\underset{\text{i}}{\mathfrak{B}}\mathbb{C}-\mathfrak{AB}+2\underset{\text{i}}{\mathfrak{C}}\mathbb{AB}-({}_{\rceil\underset{1}{BC}}A+{}_{\rceil\underset{1}{AC}}B)(1-v).$$

Problem VII. (*Milne*, 29).

To determine $(\overline{\mathfrak{AB}}\underset{\text{ii.iii}}{\mathbb{C}})$ the present value of £1 payable upon the failure of the survivor of the two lives *A, B*, provided that a third life, *C*, fail either second or last of the three.

Solution.

$$\overline{\mathfrak{AB}}\underset{\text{ii.iii}}{\mathbb{C}}=\overline{\mathfrak{AB}}-\overline{\mathfrak{AB}}\underset{\text{i}}{\mathbb{C}}.$$

But $\overline{\mathfrak{AB}}\underset{\text{i}}{\mathbb{C}}=\underset{\text{i}}{\mathfrak{C}}\mathbb{AB}+\mathfrak{A}-\mathfrak{AB}+({}_{\rceil\underset{1}{BC}}A-{}_{\rceil\underset{1}{AC}}B)(1-v)$ (Prob. IV.)

$$\therefore\ \overline{\mathfrak{AB}}\underset{\text{ii.iii}}{\mathbb{C}}=\mathfrak{B}-\underset{\text{i}}{\mathfrak{C}}\mathbb{AB}-({}_{\rceil\underset{1}{BC}}A-{}_{\rceil\underset{1}{AC}}B)(1-v).$$

Problem VIII. (*Milne*, 30; *Baily*, 50).

To determine $\left(\underset{\underset{\text{ii}}{\text{bc}}}{\underbrace{\overline{\mathfrak{AB}}}}\right)$ the present value of £1 payable upon the failure of the survivor of the two lives *A, B*, provided that *B* fail after a third life, *C*.

Solution.

$$\underset{\underset{\text{ii}}{\text{bc}}}{\underbrace{\overline{\mathfrak{AB}}}}=\underset{\text{iii}}{\mathfrak{B}}\mathbb{AC}+\underset{\text{iii}}{\mathfrak{A}}\underset{\text{ii}}{\mathbb{B}}\mathbb{C}.$$

$$\text{But } \underset{\text{III}}{\mathfrak{B}}\mathbb{A}\mathbb{C} = \mathfrak{B} - \underset{\text{I}}{\mathfrak{B}}\mathbb{A} - \underset{\text{I}}{\mathfrak{B}}\mathbb{C} + \underset{\text{I}}{\mathfrak{B}}\mathbb{A}\mathbb{C} \quad \text{(Milne, Prob. 12)}$$

$$\text{And } \underset{\text{III}}{\mathfrak{A}}\underset{\text{II}}{\mathbb{B}}\mathbb{C} = \mathfrak{A} - \underset{\text{I}}{\mathfrak{A}}\mathbb{B} - \underset{\text{I}}{\mathfrak{B}}\mathbb{A}\mathbb{C} + {}_{\rceil\underset{\text{I}}{\text{BC}}}\text{A}(1-v) \quad \text{(Prob. I.)}$$

$$\therefore \overline{\underbrace{\mathfrak{A}\mathfrak{B}}_{\underset{\text{II}}{\text{bc}}}} = \overline{\mathfrak{A}\mathfrak{B}} - \underset{\text{I}}{\mathfrak{B}}\mathbb{C} + {}_{\rceil\underset{\text{I}}{\text{BC}}}\text{A}(1-v).$$

Problem IX. (*Milne*, 31; *Baily*, 49).

To determine $\left(\overline{\underbrace{\mathfrak{A}\mathfrak{B}}_{\underset{\text{I}}{\text{bc}}}}\right)$ the present value of £1 payable upon the failure of the survivor of the two lives *A, B,* provided that *B* fail before a third life, *C.*

Solution.

$$\overline{\underbrace{\mathfrak{A}\mathfrak{B}}_{\underset{\text{I}}{\text{bc}}}} + \overline{\underbrace{\mathfrak{A}\mathfrak{B}}_{\underset{\text{II}}{\text{bc}}}} = \overline{\mathfrak{A}\mathfrak{B}}$$

$$\therefore \overline{\underbrace{\mathfrak{A}\mathfrak{B}}_{\underset{\text{I}}{\text{bc}}}} = \overline{\mathfrak{A}\mathfrak{B}} - \overline{\underbrace{\mathfrak{A}\mathfrak{B}}_{\underset{\text{II}}{\text{bc}}}}$$

$$= \underset{\text{I}}{\mathfrak{B}}\mathbb{C} - {}_{\rceil\underset{\text{I}}{\text{BC}}}\text{A}(1-v).$$

Problem X. (*Milne*, 33; *Baily*, 40).

To determine $(\underset{\text{I.III}}{\mathfrak{A}}.\underset{\text{I.III}}{\mathfrak{B}}\mathbb{C})$ the present value of £1 payable upon either *A* or *B* failing either first or last of the three lives *A, B, C.*

Solution.

$$\underset{\text{I.III}}{\mathfrak{A}}.\underset{\text{I.III}}{\mathfrak{B}}\mathbb{C} = \underset{\text{I}}{\mathfrak{A}}.\underset{\text{I}}{\mathfrak{B}}\mathbb{C} + \underset{\text{III}}{\mathfrak{A}}\underset{\text{II}}{\mathbb{B}}\mathbb{C} + \underset{\text{III}}{\mathfrak{B}}\underset{\text{II}}{\mathbb{A}}\mathbb{C}$$

$$= \underset{\text{I}}{\mathfrak{A}}.\underset{\text{I}}{\mathfrak{B}}\mathbb{C} + \overline{\mathfrak{A}\mathfrak{B}}\underset{\text{I}}{\mathbb{C}}.$$

$$\text{But } \overline{\mathfrak{A}\mathfrak{B}}\underset{\text{I}}{\mathbb{C}} = \underset{\text{I}}{\mathfrak{C}}\mathbb{A}\mathbb{B} - \mathfrak{A}\mathfrak{B} + \mathfrak{A} + ({}_{\rceil\underset{\text{I}}{\text{BC}}}\text{A} - {}_{\rceil\underset{\text{I}}{\text{AC}}}\text{B})(1-v) \quad \text{(Prob. IV.)}$$

$$\therefore \underset{\text{I.III}}{\mathfrak{A}}.\underset{\text{I.III}}{\mathfrak{B}}\mathbb{C} = \mathfrak{A} - \mathfrak{A}\mathfrak{B} + \mathfrak{A}\mathfrak{B}\mathfrak{C} + ({}_{\rceil\underset{\text{I}}{\text{BC}}}\text{A} - {}_{\rceil\underset{\text{I}}{\text{AC}}}\text{B})(1-v).$$

Substituting for ${}_{\rceil\underset{\text{I}}{\text{AC}}}\text{B}(1-v)$ its equivalent $\mathfrak{A}\mathfrak{B}\mathfrak{C} - \mathfrak{B} - {}_{\rceil\underset{\text{I}}{\text{AC}}}\text{B}(1-v)$, we have

$$\underset{\text{I.III}}{\mathfrak{A}}.\underset{\text{I.III}}{\mathfrak{B}}\mathbb{C} = \mathfrak{A} + \mathfrak{B} - \mathfrak{A}\mathfrak{B} + ({}_{\rceil\underset{\text{I}}{\text{BC}}}\text{A} + {}_{\rceil\underset{\text{I}}{\text{AC}}}\text{B})(1-v)$$

$$= \overline{\mathfrak{A}\mathfrak{B}} + ({}_{\rceil\underset{\text{I}}{\text{BC}}}\text{A} + {}_{\rceil\underset{\text{I}}{\text{AC}}}\text{B})(1-v).$$

Problem XI. (*Milne*, 34; *Baily*, 39).

To determine $(\underset{\text{II.III}}{\mathfrak{A}}.\underset{\text{II.III}}{\mathfrak{B}}\mathbb{C})$ the present value of £1 payable upon either *A* or *B* failing either second or third of the three lives *A, B, C.*

Solution.

$$\underset{\text{ii.iii}}{\mathfrak{A}}.\underset{\text{ii.iii}}{\mathfrak{B}}\mathbb{C}=\underset{\text{ii}}{\mathfrak{A}}.\underset{\text{ii}}{\mathfrak{B}}\mathbb{C}+\underset{\text{iii}}{\mathfrak{A}}\underset{\text{i}}{\mathbb{B}}\mathbb{C}+\underset{\text{iii}}{\mathfrak{B}}\underset{\text{i}}{\mathbb{A}}\mathbb{C}$$

$$\text{And }\underset{\text{i.iii}}{\mathfrak{A}}.\underset{\text{i.iii}}{\mathfrak{B}}\mathbb{C}=\underset{\text{i}}{\mathfrak{A}}.\underset{\text{i}}{\mathfrak{B}}\mathbb{C}+\underset{\text{iii}}{\mathfrak{A}}\underset{\text{ii}}{\mathbb{B}}\mathbb{C}+\underset{\text{iii}}{\mathfrak{B}}\underset{\text{ii}}{\mathbb{A}}\mathbb{C}$$

$$\therefore\ \underset{\text{ii.iii}}{\mathfrak{A}}.\underset{\text{ii.iii}}{\mathfrak{B}}\mathbb{C}+\underset{\text{i.iii}}{\mathfrak{A}}.\underset{\text{i.iii}}{\mathfrak{B}}\mathbb{C}=\underset{\text{i}}{\mathfrak{A}}.\underset{\text{i}}{\mathfrak{B}}\mathbb{C}+\underset{\text{ii}}{\mathfrak{A}}.\underset{\text{ii}}{\mathfrak{B}}\mathbb{C}+\underset{\text{iii}}{\mathfrak{A}}.\underset{\text{iii}}{\mathfrak{B}}\mathbb{C}$$

$$=\mathfrak{A}+\mathfrak{B}$$

$$\text{And }\underset{\text{ii.iii}}{\mathfrak{A}}.\underset{\text{ii.iii}}{\mathfrak{B}}\mathbb{C}=\mathfrak{A}+\mathfrak{B}-\underset{\text{i.iii}}{\mathfrak{A}}.\underset{\text{i.iii}}{\mathfrak{B}}\mathbb{C}$$

Substituting for $\underset{\text{i.iii}}{\mathfrak{A}}.\underset{\text{i.iii}}{\mathfrak{B}}\mathbb{C}$ its value as found by the last problem, we have

$$\underset{\text{ii.iii}}{\mathfrak{A}}.\underset{\text{ii.iii}}{\mathfrak{B}}\mathbb{C}=\mathfrak{A}\mathfrak{B}-({}_{\rceil_{1}^{BC}}A+{}_{\rceil_{1}^{AC}}B)(1-v).$$

Problems VI. and VII. are not contained in Baily's work, having, I believe, been first given by Milne. Many other cases of a similar nature might, doubtless, be discovered without much difficulty, and solved by the foregoing method. I will venture to add one which appears to be quite as useful as some of the preceding, and which, in the present series, will form

PROBLEM XII.

To determine ($\underset{\text{i}}{\mathfrak{A}}.\underset{\text{i}}{\mathbb{B}}\mathbb{C}$) the present value of £1 payable on the failure of a given life, *A*, provided either that life, or another given life, *B*, shall be the first that fails of the three lives *A, B, C.*

Solution.

$$\underset{\text{i}}{\mathfrak{A}}.\underset{\text{i}}{\mathbb{B}}\mathbb{C}+\underset{\text{ii iii}}{\mathfrak{A}}\mathbb{B}\underset{\text{i}}{\mathbb{C}}=\mathfrak{A}$$

$$\therefore\ \underset{\text{i}}{\mathfrak{A}}.\underset{\text{i}}{\mathbb{B}}\mathbb{C}=\mathfrak{A}-\underset{\text{ii.iii}}{\mathfrak{A}}\mathbb{B}\underset{\text{i}}{\mathbb{C}}.$$

Substituting the first value of $\underset{\text{ii.iii}}{\mathfrak{A}}\mathbb{B}\underset{\text{i}}{\mathbb{C}}$ (Prob. III.), we have

$$\underset{\text{i}}{\mathfrak{A}}.\underset{\text{i}}{\mathbb{B}}\mathbb{C}=\underset{\text{i}}{\mathfrak{A}}.\underset{\text{i}}{\mathfrak{B}}\mathbb{C}-{}_{\rceil_{1}^{BC}}A(1-v).$$

On the Rejection of the Fractions of a Pound in extensive Valuations. By PROFESSOR DE MORGAN.

WHEN a mass of results are to be calculated and added up, it generally happens that a small fraction is of little consequence in each item, except as the little of which many make a mickle. The money calculator knows nothing of the principle that it is very hard, next to impossible, practically quite impossible, to get a *mickle* out of positives and negatives, when both are sure to occur

in equal numbers in the long run, and with equal quantities. Generally speaking, the items of money calculations are determinable, and accuracy to the penny below the true result is a matter of contract. But in life contingencies, in which a certain amount of inaccuracy is known to lurk in the fundamental tables, no real truth is gained by tacking shillings, pence, and farthings, on to hundreds of pounds. If the nearest pound be always taken, granting that the calculated result shall be thereby a little altered, no man alive can say whether that alteration shall be approach to, or recession from, what we should call absolute truth, if our tables were perfect: it is as likely to be one as the other.

But do not astronomers and physical philosophers attempt the utmost accuracy of deduction in cases of which it is known that the data are erroneous? They certainly do this; but it is for the purpose of comparing the result with actual observation, in order to arrive at corrections of the data. And if ever office valuations and other assessments of results should be undertaken with a view of amending the tables of mortality by comparison of prediction and experience, I could not say a word against a degree of closeness of estimation which is now absolutely useless.

On the other hand, the astronomer and the physical* philosopher are constantly in the habit of rejecting quantities of value too minute to be worth considering; and they follow a principle which the actuary might safely adopt. When once the error of data is roughly estimated, *use* and not *correction of data* being in question, they reject all terms which are small compared with the error of the data. But an actuary who knows that, when the present value of a reversion is £1726·248 according to his tables, it cannot be told whether either £1,725 or £1,727 is not nearer the average truth of nature, nevertheless calculates his decimal places. He may be right as a *matter of business*. There must be a rule by which to award prices; and the man of business carries even into calculation the principle that by taking care of the pence the pounds will take care of themselves. And this sometimes to his

* A word is much wanted for this creature. *Physical philosopher* is unobjectionable, but too long. *Philosopher*, by itself, a word which has been used, is not only proved to be too wide, but, in the revival of philosophy proper which is taking place, will go near to be thought so shortly. *Physician* has irrevocably lost its original sense; and so has *naturalist*. *Physicist* is coming into use: it was a word invented in Pandemonium as a salute to Satan when he returned from *his* inquiry into nature, undertaken with a view of creating confusion by the introduction of his own notions; and Milton has made the natural mistake of supposing that he was *hissed*. One good distinctive word is *empiric*, follower of experience; but the physician has appropriated it to medical practitioners who are, rightly or wrongly, wise above what is written. But *pirastic* would do very well; and would prevent the experimentalist from carrying about a collar of SS.

loss, I suspect; for to take as much care of the pence as of the pounds is the same thing as taking no more care of the pounds than of the pence. Nevertheless, he would be shocked at a result given in pounds: he is prepared to believe in shillings and pence, and he asks for his full creed.

I proceed to state the result—the investigation is of too high a character—of the common theory of errors, as used in physical inquiry, applied to an addition of sums of money, each purposely made wrong by being brought up to, or down to, the nearest pound. Thus, 117·501 is made 118; but 117·499 is made 117. The accurate mean, 117·500, is always made 118: and thus all the cases from 117·000 to 117·999 are equally divided between 117 and 118. We have then errors following this law:—positive and negative errors are equally likely; the extent of error is $\pm\cdot500$, and any one error is as likely as any other. This last hypothesis is perfectly* justifiable in an actuary's calculation: in ledger items of commercial business there is a frequency of occurrence of 10*s.* 0*d.*, &c., which would require examination.

All errors then have equal chances, and the extremes are $\pm\cdot5$. The *probable* range of error, as it is called—a word of the same sort of inaccuracy as when the term within which there is even chance of death is called the *probable life*—is $\pm\cdot25$. The method which I apply substitutes, for facility of calculation, another law of error, which does not give accurate results for small numbers of cases, but very rapidly approaches the truth as the number of cases increases. The substitution will give as the probable error, not ·25, but $\cdot476936 \div \sqrt{6}$, or ·19471. For reasons given in a paper of mine on the theory of errors, lately published in the *Cambridge Transactions*, containing a second step of approximation, I hold it more accurate to increase this result in the proportion of 20 to 23, giving ·22392. The number of items in the addition being s, and $\cdot22392 \div \sqrt{s}$ being a, it is an even chance that the rejection of the decimals produces an effect intermediate between that of adding a to every item, and that of subtracting a from every item. Thus, if s be 100, and if in these 100 cases the

* I suppose that such a formula as $(1-rA):(1+r)$ must needs present all digits equally in the long run, there being no law in the values of A as to digits. But no such thing is to be inferred of numbers which are not accidentally arranged. In Mr. Shanks's ratio of the circumference of a circle to its diameter, or $4(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots)$, to 608 figures, the nine digits and cipher occur as follows:—

7, 44 times	8, 58 times	1, 62 times	4, 64 times	9, 67 times
5, 56 times	0, 60 times	6, 62 times	2, 67 times	3, 68 times

The exceptional character of the 7 is remarkable.

nearest pound be taken, it is an even chance that the effect produced is not so great as would be produced by altering each item, in one direction, by $5\frac{1}{2}d.$; or, be the whole effect augmentation, or be it diminution, as likely as not it will not amount to $5\frac{1}{2}d.$ per item.

Now take the following table :—

k.	*l.*	*k.*	*l.*	*k.*	*l.*	*k.*	*l.*
1·1	·542	2·1	·843	3·1	·963	4·1	·99431
1·2	·582	2·2	·862	3·2	·969	4·2	·99539
1·3	·619	2·3	·879	3·3	·974	4·3	·99627
1·4	·655	2·4	·895	3·4	·978	4·4	·99700
1·5	·688	2·5	·908	3·5	·982	4·5	·99760
1·6	·719	2·6	·921	3·6	·985	4·6	·99808
1·7	·748	2·7	·931	3·7	·987	4·7	·99848
1·8	·775	2·8	·941	3·8	·990	4·8	·99879
1·9	·800	2·9	·950	3·9	·991	4·9	·99905
2·0	·823	3·0	·957	4·0	·993	5·0	·99926

The meaning of this table is as follows :—The result a being as above, then l is the probability that the effect of the error will not be that of ka added to every item, or subtracted from every item. Thus it is ·99926 to ·00074, or 1350 to 1, against reduction to the nearest pound producing as much effect upon the summation of 100 items as a change of $5 \times 5\frac{1}{2}d.$, or $2s.\ 3\frac{1}{2}d.$, in each item. Of course I need hardly say that nothing but multiplication or division of the result by 10 is needed for the case where the nearest 10 pounds, or the nearest tenth of a pound, is taken.

Perhaps some may be induced to verify the preceding statements. As all matters connected with probability are of interest to your readers, I may here give a more complete account than I have given elsewhere of the verifications of the celebrated *Petersburg Problem.*

This problem is an old paradox, which was once thought to throw doubt on the theory. Let a halfpenny be thrown until head appears; and let this succession of throws, be it H, or TH, or TTH, &c., be called a *set*. If the set contain n throws, 2^n pounds is to be paid by the banker: how much ought he to receive to stand the risk of one set? The answer is instantly seen to be $\frac{1}{2}\times 2+\frac{1}{4}\times 4+\frac{1}{8}\times 8+ \ldots.$ or $1+1+1+\ldots.$ which is infinite. The answer is true; that is, no sum per set is great enough to meet the long run. The long sets, though they occur rarely, yield in proportion to their rarity, and thus do as much towards raising

the average value of a set as the shorter ones. Buffon tried 2,048 sets; and in our own day three others, whom I shall call H, P, S, have repeated his mode of trial. The first column shows the throw at which head first occurred; the second shows the result of theory, as more likely than any one other disposition; the next columns show the four trials; the last column but one, the sum of all four; and the last shows what theory would predict on that sum.

		B.	H.	P.	S.		
1	1,024	1,061	1,048	1,017	1,039	4,165	4,096
2	512	494	507	547	480	2,028	2,048
3	256	232	248	235	267	982	1,024
4	128	137	99	118	126	480	512
5	64	56	71	72	67	266	256
6	32	29	38	32	33	132	128
7	16	25	17	10	19	71	64
8	8	8	9	9	10	36	32
9	4	6	5	3	3	17	16
10	2	0	3	2	4	9	8
11	1	0	1	1	0	2	4
12		0	0	1	0	1	2
13		0	0	0	0	0	1
14	1	0	1	0	0	1	
15		0	0	0	0	0	1
16		0	1	1	0	2	
&c.							
	2,048	2,048	2,048	2,048	2,048	8,192	8,192

Thus Buffon saw head deferred until the ninth throw six times, but nothing further. H and P both saw it deferred until the sixteenth throw. These instances illustrate the theorem that all which can happen will happen at last, if trials enough be made; and also that two similar sets of trials may show very different amounts of difficult cases. Thus B and S would probably have concurred in the conclusion that it is useless to look for very remote single accidents, and that whatever is to be looked for will come in plural numbers; but H and P would know better. If anyone will calculate the amount that either would have received, one throw with another, on the conditions of the game, he will see that the banker would need a much larger capital than unassisted judgment would suppose. One deferment of head to the sixteenth throw, by itself, would realise £32 per set to the opponent, through all the 2,048 sets.

On the Valuation of Policies of Assurance.

Policies of life assurance are of many kinds—subject to the payment of premiums in various ways, and to risks of different descriptions.

A policy has several values, according to the purposes to which it is applied, the necessities of the holder, and the intention of the purchaser.

The following pages are intended as a guide to those who, not having made the nature and uses of a policy a subject of study, may yet have occasion to form an estimate of its value for some special purpose.

The complete investigation of every possible case would be impracticable; a whole-life policy, subject to the payment of annual premiums on the non-participating scale, will, therefore, be considered in its various relations as illustrative of the methods to be adopted in dealing with others; and a variety of cases will then be adduced, showing how policies of other kinds may be valued. The further investigation of the many varieties of policy and modes of premium payment adopted in the present day, frequently requiring mathematical attainments of a high order, belongs properly to the actuary, who should be consulted in all cases of difficulty.

A whole-life policy, on the non-participating scale and subject to the payment of a fixed annual premium, has several independent values,* according to the purposes for which, and the circumstances under which, the valuation is made—each value depending on the use to which the policy is to be applied.

Thus, to the assured himself, the policy has a money value, depending on the rate of premium, the number of payments already made, and the current rate of interest at which money can be invested or borrowed.

To the lender, who has accepted a life policy as security for the repayment of a loan, it has a money value depending on the same elements as above, but further depreciated by the risk of the assured discontinuing to pay the premiums or interest, and of his going abroad or in any way vitiating the policy.

In the open market a policy has a value depending, in some degree, upon the estimate the public make of the character and

* The moral worth of a policy it is not presumed to estimate, for it is inestimable. The question of money value alone is treated of in these pages, because it alone is marketable.

stability of the assuring Office; also on the presumed state of health and the habits of the assured, and other circumstances peculiar to each case.

In connexion with life interests and reversions, a policy has, in the market, other values, which will be exemplified; and

The surrender value of a policy differs from all, varying according to the rules of the several Offices.

Before proceeding to the consideration of these several values and the modes of estimating them, it will be well to examine the Rules usually given.

A standard writer says:—

"When the premium is just due and not paid, add unity to the present value of the annuity on the life at the time of disposing of the policy, multiply it by the annual premium payable on the policy, and subtract the product from the single premium which would be charged for insuring the sum at the present age of the life in the policy." [*Rule* A.]

"Or,—Take the difference between the premium which would be required at the present age and the premium charged in the policy, multiply it by unity added to the value of the annuity at the present age of the life in the policy." [*Rule* B.]

"Or,—Increase by unity the value of an annuity of £1 at the present age, and divide the sum by unity added to the present value of an annuity of £1 at the age when the policy was effected, substract the quotient from unity, and multiply the difference by the sum assured." "This Rule applies only when the annual premium has been calculated at the same rate per cent. and the same table of mortality as are used in valuing the policy." [*Rule* C.]

Applying these Rules to the case of a policy for £1,000 on a life now aged 60, effected in a respectable Office 40 years ago, at an annual premium of £17. 5*s*. 10*d*.—a similar policy taken out now requiring in the same Office a premium of £67. 1*s*. 7*d*., and the annuity table of the same Office charging, at age 20, 21·7014 years' purchase, and at age 60, 11·0656 years' purchase—the following three values result:—

Rule A gives . . .	£488	17	9
" B " . . .	600	14	11
" C " . . .	468	10	2

It is therefore evident that these Rules are not of equal value, nor applicable to one and the same purpose.

Rule A amounts to this—that, in making the valuation, the cost of an assurance of £1,000 at age 60, by a single premium, is to be accepted as the value of the £1,000 in reversion; and that the

value of the policy is worth just so much, minus the cost of the annuity due, of the premiums actually payable. This Rule gives the value of the policy to a person who has only the option of assuring £1,000 on the life by a single premium, or of taking an assignment of the existing policy. But this option assumes the willingness of the Office to accept the life and to grant a policy at the present time; and the wish to make the investment on such terms implies, on the part of the buyer, a notion that the life is not so good as an average one of the age—otherwise he may look forward to finding his purchase a bad investment, through the prolonged life of the assured, against which this mode of valuation makes no provision.

Rule B gives the difference between the cost of an annuity due, of the premium actually payable, and of that which would be payable if the Office would grant a whole-life policy on the life of the assured at his present age. This Rule gives, therefore, the value of the policy to a person who has the option of taking an assignment of it or of effecting another policy on the same life in the same Office, and purchasing an annuity of the premium; but it assumes that no other option exists; it assumes a *necessity* for the transaction and for the precise mode of carrying it out. No one who could procure the assurance on the life by the payment of a single premium of £697. 10*s.* would purchase an annuity of the annual premium of £809. 7*s.* 2*d.* This Rule is, by some, supposed to give the surrender value, or price the Office should allow for the policy; but such is not the case, as will be shown under the head of "Surrender Value."

Rule C is confessedly based on theoretical values; it can therefore be true only on the assumption that net premiums are charged, and the same rate of interest and table of mortality are used, in the calculation of the annuity and premium values and of the policy.

It is clear, therefore, that neither of these Rules applies to the cases occurring in actual practice — neither of them recognising the fact, that a person purchasing a policy as an investment requires a larger rate of interest than any Office allows on the premiums paid, and each of them assuming conditions which are neither essential nor probable.

The question must, therefore, be considered *de novo,* in its various phases.

CASE 1.—*The value, to the assured, of a whole-life policy, subject to the payment of annual premiums of a fixed amount.*

This case is divisible into three, for the assured may—

1*a*. Retain the policy, running the risk of being able to keep it in force.

1*b*. Retain it, providing for the future payment of premiums by the purchase of a sufficient annuity on his own life. Or,

1*c*. He may part with it, converting it into cash, by surrender or sale.

CASE 1*a*.—The assured retaining the policy, at the risk of making or failing to make future premium payments.

The monetary value of this policy to the assured (even if his means are such that no measurable risk attaches) is generally less than nothing, because the case supposes the assured to retain the policy, making all necessary payments, and deriving no pecuniary benefit; and, if his income be precarious, depending on his continued exertions and the necessary maintenance of his health, the only value such policy has, is to be found in the satisfaction and ease of mind derived from the *notion* that some provision is thus made for others in the event of premature death, or for himself (by surrender or sale) in illness or on superannuation. There is a case in which such a policy has a pecuniary value, but it is then only in connexion with other matters, and not *per se;* it may enable a man to employ capital derived from a source of such a kind that, at his death, it must be replaced, and the policy then becomes indirectly valuable, its worth being measured by the use that is made of the capital so temporarily employed.

CASE 1*b*.—The assured retaining the policy in trust for another (or giving it away), all future charges on it being provided for by the purchase of an annuity.

In this case the policy becomes a secure investment, but its worth must be estimated in accordance with the object in view in the valuation; thus

If it be necessary to ascertain the value of the assets of the assured, for any present purpose, the policy must be estimated at its selling value. (See below.)

If the policy be considered as part of a property to be devised by will, it should be taken at its full value as at the time when it becomes a claim.

If the valuation has for its object an immediate partition of

property, of which the policy forms part, its value must be found by one of the following Rules.

If the annuity of the premium is bought at the expense of the assured, before partition, and applied to the payment of future premiums, the policy is worth, in the partition, precisely as much as an absolute reversion to the sum assured, which is found thus:—

From the amount assured, deduct the actual cost of an annuity due, upon the life of the assured, of one year's discount. [*Rule* D.]

This value is found thus:—Taking the policy before referred to,

From the sum assured . . .	£1,000
Deduct an annuity due of the discount, viz., $47\cdot619 \times 12\cdot0656 =$ . .	574·55
	£425·45

If, in the partition, the person receiving the policy is charged with the purchase of the annuity requisite for the payment of future premiums, it will be right to deduct from the value just found, the cost of such annuity. [*Rule* E.]

From	£425·45
Deduct $17\cdot29 \times 12\cdot0656 =$ . .	268·614
	£216·836

Case 1*c*.—This falls under the heads of "Surrender" and "Market Values," which see.

Case 2.—*A life policy held by a creditor of the assured.*

This case is divisible, for—

2*a*. The policy may have been surrendered to the creditor in part satisfaction of a debt, and assigned absolutely, in which event it is worth precisely the sum for which it will sell, although circumstances may induce the creditor to hold it and continue the payment of premiums.

2*b*. It may be held merely as a security for the payment of a debt and interest, in which case it never can be worth, to the creditor, more than the amount owing, because the assured has in it a legally recognised interest.

In order to ascertain how far the policy is a sufficient security, the creditor must deduct, from the value of the reversion, the cost of an annuity of the premiums and of the interest on the debt.

CASE 2*b*.—In this case there are two valid interests, viz., that of the creditor and that of the debtor.

The creditor's interest in the policy is limited to the amount of the debt and interest at the time of valuation.

The assured's interest at any given time is the difference between the value deduced by Rule E and the amount of his debt, with the cost of an annuity of the interest during his life added.

Thus, if the policy already referred to be held as security for a debt of £100, and if it be required to determine whether, at the present time (a year's interest at 5 per cent. being just due), it be a sufficient security, and what the surplus is in favour of the debtor, the calculation will be as follows:—

From the value by Rule E		£216·836
Deduct—The debt . . .	£100	
The cost of an annuity due, of £5 per annum, on the life of the assured, *i. e.*, 12·0656×5= . . .	60·328	
		160·328
		£56·508

The surplus shows the security to be sufficient, and that the assured's interest in the policy is £56. 10*s*. 2*d*.

The rule for finding this surplus may be thus expressed:—

From the present worth of a reversion to the sum assured (found by Rule D, or immediately by the use of my *Tables*, p. 93), deduct the debt and the cost of an annuity due, on the assured's life, of the premiums and interest. [*Rule* F.]

Or,—From the sum assured, deduct the debt and the cost of an annuity due, of the (1) discount on the sum assured, (2) the premium, and (3) the interest on the debt. [*Rule* G.]

This rule is applicable when the debtor declines to give up his legal right in the policy, and cannot continue the payment of premiums or interest. The creditor may safely hold the policy as a security, and continue making the necessary premium payments, until the assured can redeem it by payment of the debt and all arrears of interest, premiums and charges. The values of the two interests in this policy vary from year to year as the value of the policy increases, and that of the debt grows by the accumulation of added premiums paid and interest in arrears.

Case 3.—*Marketable value.*

When a policy is offered for public sale, the amount that may be expected in realization depends on the state of the money market, as well as upon general principles; and a policy in a popular Office will readily sell, while others find no bidder.

A common, but erroneous, mode of valuation in this case is, to ascertain from the ordinary tables of reversions at 6 per cent. (calculated on the assumption that 6 per cent. can be obtained at all times, and that money can, therefore, be invested at any time you please at that rate), the value of a reversion to the sum assured, and deduct the worth of an annuity of the premium by the Northampton Table at 3 per cent., thus—

Reversion to £1,000, after a life aged 60, by Northampton Table 6 per cent., worth .	£500·75
Deduct value of annuity due, of the premium, by 3 per cent. Table—*i. e.*, 10·7774 years' purchase, or	186·34
	£314·41

A guess would be about as accurate and as trustworthy as this rude estimate, not one element of which is of any worth.

The purchaser of a life policy should, before bidding for it—before placing himself in a position in which he may be tempted to make an advance upon the decision of his judgment—ascertain, as accurately as existing data will allow, the real value of the document he wishes to possess. For this purpose he should learn the financial position of the Office in which the policy is effected, and the general character of its management; and, if he be unwilling or unable to avail himself of the assistance of an actuary conversant with such matters, he may, at least, ascertain the opinion of those who trade in shares, by consulting the quoted prices in a share list.

If he be satisfied that the Office is sound and well managed, he may treat the policy as a valid security for all that the Office, on its part, engages. If he have reason to doubt the soundness of the Office, he must determine for himself the extent of risk to which he is willing to run, and at what rate of interest the calculation shall be made to meet the risk. He may then obtain a valuation as follows:—

Case 3*a*.—To secure 5 per cent. per annum on the outlay, purchasing an annuity of the premiums and interest in the Govern-

ment Offices (the Assurance Office being considered safe, and a premium just due).

To the annual premium payable . .	£17·29
Add a years' discount, at 5 per cent., on the amount assured	47·62
The annuity to be purchased is . .	64·91
Multiply by the present worth of an annuity due, at the present age of the assured, according to the Government tables, having regard to the price of Consols; if these are at 95, the price of an annuity at age 60 is 11·52 years' purchase; an annuity due is to be taken as . .	12·52 years' purchase;
And the present worth of all the premiums and interest, at 5 per cent., to be deducted from the sum assured, is, therefore .	£812·675
Leaving the real value of the policy *on the data* indicated	£187·325, or £187 6 6

Case 3*b*.—To secure 5 per cent. per annum on the outlay, purchasing an annuity of the premiums and interest in an Office selected by the intended purchaser, in which the charge for such annuity is (say) 9·952 years' purchase.

The valuer, in this case, does not, of course, undertake any responsibility with regard to the safety of the Office in which the annuity is to be purchased; he merely makes his valuation on the data supplied, and should guard his report accordingly.

Annual premium and discount, as above .	64·91
Multiply by price of annuity due . .	10·952
Present worth of all premiums and interest	£710·9
The price that, under the circumstances, may be given, is	£289·1, or £289 2 0

Case 3*c*.—To promise 8 per cent., the purchaser accepting the risk of an indifferent Office, and considering himself safe in making a deduction for the annuity at the rate indicated by the Carlisle Tables at $3\frac{1}{2}$ per cent.

Annual premium	£17·29
Discount on £1,000 for one year, at 8 per cent.	74·075
	91·365
Worth of annuity due, by Carlisle Table at 3½ per cent.	11·063
The product of these, or total deduction to be made from the sum assured in order to meet the case	£1010·77

The charges for risk and premium exceeding the gross value of the sum assured, the valuer will be only too glad to say that, under the circumstances, the policy cannot be purchased at a price that will meet his client's requirements.

It thus appears, very clearly, that the price any person may be willing to give for a policy on another's life, offered for sale in the market, depends, or should depend, on the mode in which he purposes making his investment safe; and, although a Rule may be given definitely, its application must vary with individual opinion as to the data to be employed; and these should always be distinctly expressed as between a valuer and the person for whom he acts.

The Rule, when a premium is just due, is—

From the sum assured deduct the price at which an annuity due, of the premium payable and discount (on the gross sum assured), can be obtained in an Office or manner satisfactory to the buyer. [*Rule* H.]

The limits within which an annuity on a life aged 60 may be purchased are very wide, varying, in the tables of different Offices lying before me (March, 1862) from 9·28 to 12·57 years' purchase. The limits within which purchasers willing to take the policy in question, to secure 5 per cent. on their outlay, may vary their estimates, are, therefore, £119. 4*s*. and £332. 14*s*.

Case 4.—*The value of a policy offered for sale in connexion with a life interest.*

In this case the vendor's interest in the property put into the market being only occurrent with his life, the policy offered for sale is to be considered as *quantum valeat* security to the purchaser, as against the vendor's premature death. But, unless its value is equivalent to the risk, or sufficient to indemnify the purchaser entirely, it will be necessary for him to further secure himself;

and, if its value be more than enough for the purpose, the purchaser should pay an additional price for the surplus.

The policy may be either—

1st. Exactly sufficient to form a perfect security.
2ndly. Insufficient.
3rdly. More than sufficient.

The first question to be solved is, What sum should be assured in order to cover the costs of purchase of a life estate? The proper mode of finding this is, to ascertain the rate of annual premium charged for the assurance of £1 on the life, to add to this the discount on £1 for a year at the rate of interest required, and to divide the annual income, rent or annuity, by the sum. [*Rule* I.]

If the amount thus found, and the sum assured in the policy offered for sale, correspond (an event scarcely probable in ordinary practice), the life interest and policy together are worth this amount discounted for a year.

Thus, if, just after a premium payment, the policy to which reference has been made were offered for sale with the assured's life interest in an annuity of £64. 18*s*. 2*d*. per annum, its sufficiency, in order to secure the purchaser 5 per cent. interest, will be found as follows:—

The premium charged on £1,000 being 17·29, that for £1 is	·01729
The discount on £1 for a year, at 5 per cent.	·047619
Their sum . . .	·064909

Dividing the annuity by this sum, it is seen that the result agrees with the amount assured (£1,000); and this being discounted for a year, the price that may be given with safety for the life interest and policy together is £952. 7*s*. 7½*d*., which includes one premium repaid to the vendor.

2ndly. If the annuity were for £90 per annum, it is evident that the policy is insufficient.

The purchaser may, in this case, give for the £64. 18*s*. 2*d*. per annum covered by the policy, the sum found above—viz., £952. 7*s*. 7½*d*.; and must purchase the remainder of the annuity—viz., £25. 1*s*. 10*d*. per annum—at such a price as will enable him to secure himself by a fresh policy on the annuitant's life. To find the sum he can afford to give for this portion of the annuity, he must—

Add to the premium *now* chargeable for £1 upon the annuitant's life, the discount on £1 for a year; divide the annuity by the sum, and deduct one year's value of the annuity; thus—

Premium now chargeable	·06708
Discount on £1 for one year . . .	·047619
	·114699

Dividing the annuity, £25·092, by this, and deducting £25·092 from the quotient, the sum to be given for this extra portion is found to be £193. 13*s.* 3*d.*

And the total purchase price of the £90 a year, with the policy for £1,000, is £1,146. 0*s.* 11*d.*

3rdly. The policy may be of more than sufficient value to cover the annuity.

For the annuity might be for £50 a year, in which case a policy for £770. 6*s.* 2*d.*, subject to the premium actually paid, would have sufficed (£50 divided by ·064909 being = 770·309).

In this case, that portion of the policy by which the amount assured is in excess of the sum absolutely necessary has a value to be estimated as that of an absolute reversion. Thus—

From the sum assured . . .	£1,000
Deduct the amount necessary to secure the annuity of £50, found as above .	770·309
Absolute reversion to be valued . .	£229·691
By Rule D this is worth . . .	·42545 per £1.
Multiply this, the reversion is found to be worth	£97·722

The annuity for £50, thus amply secured, may, therefore, be purchased to pay 5 per cent.; for—

£770·309 discounted for a year= . .	733·628
And the excess of the policy for . .	07.722
Making together . .	£831·350

Table of Life Annuities, in connexion with Policies of Assurance, to allow the Purchaser a Net Interest of 5 per Cent.

The sum for which the policy should be effected, to cover an annuity of £1, is 1 more than the value of the annuity unsecured, or $\frac{1}{p+d}$.

$100p$ Premium charged per Cent.	$\frac{1}{p+d}-1$ Worth of Annuity in years' purchase, the Buyer effecting the Assurance.	$\frac{v}{p+d}$ Worth of the Annuity when secured by the Policy.	$p+d$ Annuity covered by every £1 assured.	$100p$ Premium charged per Cent.	$\frac{1}{p+d}-1$ Worth of Annuity in years' purchase, the Buyer effecting the Assurance.	$\frac{v}{p+d}$ Worth of the Annuity when secured by the Policy.	$p+d$ Annuity covered by every £1 assured.
£ s. d.				£ s. d.			
1 0 0	16·355	16·529	·057 619	1 11 9	14·750	15·000	·063 494
1 0 3	·318	·493	744	1 12 0	·719	14·970	619
1 0 6	·280	·458	869	1 12 3	·688	·941	744
1 0 9	·243	·422	994	1 12 6	·657	·911	869
1 1 0	·206	·387	·058 119	1 12 9	·626	·882	994
1 1 3	·169	·352	244	1 13 0	·596	·853	·064 119
1 1 6	·132	·317	369	1 13 3	·566	·824	244
1 1 9	·096	·282	494	1 13 6	·535	·796	369
1 2 0	·059	·247	619	1 13 9	·505	·767	494
1 2 3	·023	·212	744	1 14 0	·475	.738	619
1 2 6	15·987	·178	869	1 14 3	·445	·710	744
1 2 9	·951	·144	994	1 14 6	·416	·682	869
1 3 0	·915	·110	·059 119	1 14 9	·386	·653	994
1 3 3	·879	·076	244	1 15 0	·356	·625	·065 119
1 3 6	·844	·042	369	1 15 3	·327	·597	244
1 3 9	·808	·008	494	1 15 6	·298	·569	369
1 4 0	·773	15·974	619	1 15 9	·269	·541	494
1 4 3	·738	·941	744	1 16 0	·239	·514	619
1 4 6	·703	·908	869	1 16 3	·211	·486	744
1 4 9	·668	·875	994	1 16 6	·182	·459	869
1 5 0	·634	·842	·060 119	1 16 9	·153	·431	994
1 5 3	·599	·809	244	1 17 0	·124	·404	·066 119
1 5 6	·566	·776	369	1 17 3	·096	·377	244
1 5 9	·531	·743	494	1 17 6	·067	·350	369
1 6 0	·497	·711	619	1 17 9	·039	·323	494
1 6 3	·463	·679	744	1 18 0	·011	·296	619
1 6 6	·429	·646	869	1 18 3	13·983	·269	744
1 6 9	·395	·614	994	1 18 6	·955	·242	869
1 7 0	·362	·582	·061 119	1 18 9	·927	·216	994
1 7 3	·328	·551	244	1 19 0	·899	·189	·067 119
1 7 6	·295	·519	369	1 19 3	·871	·163	244
1 7 9	·262	·487	494	1 19 6	·844	·137	369
1 8 0	·229	·456	619	1 19 9	·816	·111	494
1 8 3	·196	·425	744	2 0 0	·789	·085	619
1 8 6	·163	·393	869	2 0 6	·734	·033	869
1 8 9	·131	·362	994	2 1 0	·680	13·981	·068 119
1 9 0	·098	·332	·062 119	2 1 6	·627	·930	369
1 9 3	·066	·301	244	2 2 0	·573	·879	619
1 9 6	·034	·270	369	2 2 6	·520	·829	869
1 9 9	·002	·240	494	2 3 0	·468	·779	·069 119
1 10 0	14·970	·209	619	2 3 6	·427	·740	369
1 10 3	·938	·179	744	2 4 0	·364	·680	619
1 10 6	·906	·149	869	2 4 6	·312	·631	869
1 10 9	·875	·119	994	2 5 0	·261	·582	·070 119
1 11 0	·843	·089	·063 119	2 5 6	·211	·534	369
1 11 3	·812	·059	244	2 6 0	·160	·486	619
1 11 6	·781	·029	369	2 6 6	·111	·439	869

Table of Life Annuities (continued).

Premium per Cent. £	s.	d.	Worth of Annuity unsecured.	Worth of Annuity secured.	Annuity covered by each £1 Assured.	Premium per Cent. £	s.	d.	Worth of Annuity unsecured.	Worth of Annuity secured.	Annuity covered by each £1 Assured.
2	7	0	13·061	13·391	·071 119	3	15	0	10·748	11·189	·085 119
2	7	6	·012	·344	369	3	15	6	·714	·156	369
2	8	0	12·963	·298	619	3	16	0	·680	·128	619
2	8	6	·914	·252	869	3	16	6	·646	·091	869
2	9	0	·866	·206	·072 119	3	17	0	·612	·059	·086 119
2	9	6	·818	·160	369	3	17	6	·578	·027	369
2	10	0	·770	·115	619	3	18	0	·545	10·995	619
2	10	6	·723	·070	869	3	18	6	·512	·963	869
2	11	0	·676	·025	·073 119	3	19	0	·479	·932	·087 119
2	11	6	·630	12·981	369	3	19	6	·446	·901	369
2	12	0	·583	·937	619	4	0	0	·413	·870	619
2	12	6	·537	·893	869	4	0	6	·381	·839	869
2	13	0	·492	·849	·074 119	4	1	0	·348	·808	·088 119
2	13	6	·446	·806	369	4	1	6	·316	·777	369
2	14	0	·401	·763	619	4	2	0	·284	·747	619
2	14	6	·357	·721	869	4	2	6	·253	·717	869
2	15	0	·312	·678	·075 119	4	3	0	·221	·687	·089 119
2	15	6	·268	·636	369	4	3	6	·190	·657	369
2	16	0	·224	·594	619	4	4	0	·158	·627	619
2	16	6	·181	·553	869	4	4	6	·127	·597	869
2	17	0	·137	·512	·076 119	4	5	0	·096	·568	·090 119
2	17	6	·094	·471	369	4	5	6	·066	·539	369
2	18	0	·052	·430	619	4	6	0	·035	·510	619
2	18	6	·009	·390	869	4	6	6	·005	·481	869
2	19	0	11·967	·349	·077 119	4	7	0	9·975	·452	·091 119
2	19	6	·926	·310	369	4	7	6	·945	·423	369
3	0	0	·884	·270	619	4	8	0	·915	·395	619
3	0	6	·842	·231	869	4	8	6	·885	·367	869
3	1	0	·801	·191	·078 119	4	9	0	·856	·339	·092 119
3	1	6	·760	·153	369	4	9	6	·826	·311	369
3	2	0	·720	·114	619	4	10	0	·797	·283	619
3	2	6	·679	·075	869	4	10	6	·768	·255	869
3	3	0	·639	·037	·079 119	4	11	0	·739	·228	·093 119
3	3	6	·599	11·999	369	4	11	6	·710	·200	369
3	4	0	·560	·962	619	4	12	0	·682	·173	619
3	4	6	·520	·924	869	4	12	6	·653	·146	869
3	5	0	·481	·887	·080 119	4	13	0	·625	·119	·094 119
3	5	6	·443	·850	369	4	13	6	·597	·092	369
3	6	0	·404	·813	619	4	14	0	·569	·065	619
3	6	6	·366	·777	869	4	14	6	·541	·039	869
3	7	0	·328	·741	·081 119	4	15	0	·513	·013	·095 119
3	7	6	·290	·704	369	4	15	6	·486	9·986	369
3	8	0	·252	·669	619	4	16	0	·458	·960	619
3	8	6	·215	·633	869	4	16	6	·431	·934	869
3	9	0	·177	·598	·082 119	4	17	0	·404	·908	·096 119
3	9	6	·146	·562	369	4	17	6	·377	·883	369
3	10	0	·104	·527	619	4	18	0	·350	·857	619
3	10	6	·067	·493	869	4	18	6	·323	·832	869
3	11	0	·031	·458	·083 119	4	19	0	·297	·806	·097 119
3	11	6	10·995	·424	369	4	19	6	·270	·781	369
3	12	0	·959	·390	619	5	0	0	·244	·756	619
3	12	6	·923	·356	869	5	0	6	·218	·731	869
3	13	0	·888	·322	·084 119	5	1	0	·192	·706	·098 119
3	13	6	·853	·288	369	5	1	6	·166	·682	369
3	14	0	·818	·255	619	5	2	0	·140	·657	619
3	14	6	·783	·222	869	5	2	6	·114	·633	869

Table of Life Annuities (continued).

Premium per Cent.			Worth of Annuity unsecured.	Worth of Annuity secured.	Annuity covered by each £1 Assured.	Premium per Cent.			Worth of Annuity unsecured.	Worth of Annuity secured.	Annuity covered by each £1 Assured.
£	s.	d.				£	s.	d.			
5	3	0	9·089	9·608	·099 119	7	2	0	7·430	8·029	·118 619
5	3	6	·063	·584	369	7	3	0	·395	7·995	·119 119
5	4	0	·038	·560	619	7	4	0	·360	·962	619
5	4	6	·013	·536	869	7	5	0	·325	·929	·120 119
5	5	0	8·988	·512	·100 119	7	6	0	·291	·896	619
5	5	6	·963	·489	369	7	7	0	·256	·863	·121 119
5	6	0	·938	·465	619	7	8	0	·222	·831	619
5	6	6	·914	·442	869	7	9	0	·189	·799	·122 119
5	7	0	·889	·418	·101 119	7	10	0	·155	·767	619
5	7	6	·865	·395	369	7	11	0	·122	·735	·123 119
5	8	0	·841	·372	619	7	12	0	·089	·704	619
5	8	6	·817	·349	869	7	13	0	·057	·673	·124 119
5	9	0	·792	·326	·102 119	7	14	0	·024	·642	619
5	9	6	·769	·303	369	7	15	0	6·992	·612	·125 119
5	10	0	·745	·281	619	7	16	0	·961	·582	619
5	10	6	·721	·258	869	7	17	0	·929	·551	·126 119
5	11	0	·698	·236	·103 119	7	18	0	·989	·522	619
5	11	6	·674	·213	369	7	19	0	·867	·492	·127 119
5	12	0	·651	·191	619	8	0	0	·836	·463	619
5	12	6	·628	·169	869	8	1	0	·805	·434	·128 119
5	13	0	·604	·147	·104 119	8	2	0	·775	·405	619
5	13	6	·581	·125	369	8	3	0	·745	·376	·129 119
5	14	0	·558	·103	619	8	4	0	·715	·348	619
5	14	6	·536	·082	869	8	5	0	·685	·319	·130 119
5	15	0	·513	·060	·105 119	8	6	0	·656	·291	619
5	15	6	·490	·039	369	8	7	0	·627	·263	·131 119
5	16	0	·468	·017	619	8	8	0	·598	·236	619
5	16	6	·446	8·996	869	8	9	0	·569	·209	·132 119
5	17	0	·423	·975	·106 119	8	10	0	·540	·181	619
5	17	6	·401	·954	369	8	11	0	·512	·154	·133 119
5	18	0	·379	·933	619	8	12	0	·484	·128	619
5	18	6	·357	·912	869	8	13	0	·456	·101	·134 119
5	19	0	·335	·891	·107 119	8	14	0	·428	·075	619
5	19	6	·314	·870	369	8	15	0	·401	·048	·135 119
6	0	0	·292	·850	619	8	16	0	·374	·022	619
6	1	0	·249	·809	·108 119	8	17	0	·347	6·997	·136 119
6	2	0	·206	·768	619	8	18	0	·320	·971	619
6	3	0	·164	·728	·109 119	8	19	0	·293	·946	·137 119
6	4	0	·123	·688	619	9	0	0	·266	·920	619
6	5	0	·081	·649	·110 119	9	1	0	·240	·895	·138 119
6	6	0	·040	·610	619	9	2	0	·214	·870	619
6	7	0	7·999	·571	·111 119	9	3	0	·188	·846	·139 119
6	8	0	·959	·532	619	9	4	0	·162	·821	619
6	9	0	·919	·494	·112 119	9	5	0	·137	·797	·140 119
6	10	0	·879	·457	619	9	6	0	·111	·773	619
6	11	0	·840	·419	·113 119	9	7	0	·086	·749	·141 119
6	12	0	·801	·382	619	9	8	0	·061	·725	619
6	13	0	·763	·346	·114 119	9	9	0	·036	·701	·142 119
6	14	0	·725	·309	619	9	10	0	·012	·678	619
6	15	0	·687	·273	·115 119	9	11	0	5·987	·654	·143 119
6	16	0	·649	·237	619	9	12	0	·963	·631	619
6	17	0	·612	·202	·116 119	9	13	0	·939	·608	·144 119
6	18	0	·575	·167	619	9	14	0	·915	·585	619
6	19	0	·538	·132	·117 119	9	16	0	·867	·540	·145 619
7	0	0	·502	·097	619	9	18	0	·820	·496	·146 619
7	1	0	·466	·063	·118 119	10	0	0	·774	·452	·147 619

Case 5.—*The value of a policy in connexion with a contingent reversion.*

The object of the policy, in this case, is to secure the purchaser of the reversion against an adverse contingency: thus, if A is possessed of a life interest in an estate, and B is entitled to the reversion in the event of his surviving A, B cannot make his interest absolute, in order to effect a sale, without securing the purchaser against loss in the event of his dying before A; but he may do this by having recourse to a Life Assurance Office.

The proper amount for which the policy should be effected is, the worth of the estate as if in possession; and the vendor should, before offering the reversion for sale, be careful to cover the whole by assurance, in order to give the property a marketable value.

But it may be that an insufficient policy has been effected some years ago, and that the reversion is offered for sale only partially secured; in which case, that portion of the reversion which is covered—viz., the amount assured under the policy, may be valued as follows:—

From the gross sum assured, deduct the cost of an annuity due (on the joint life of the life-holder and reversion-expectant) of the premium charged in the policy and the discount for one year on the sum assured. [*Rule* K.]

The value of the remaining portion of the reversion depends on the possibility of effecting an assurance, and on the rate at which the assurance can be effected. If the assurance is practicable, this portion of the property may be valued by the same Rule as the other, using, instead of the premium charged in the existing policy, that which must be paid on the new one; or the two valuations may be made together, thus:—

From the gross value of the estate, deduct the cost of an annuity due (on the joint life) for the amount of the two premiums and the discount on the gross value for one year. [*Rule* L.]

Thus, B, aged 35, is entitled, in reversion (contingent on his surviving A, aged 55), to an estate worth £10,000; and offers his contingent interest for sale, together with a policy for £6,000, effected 20 years ago on his life against A's, at an annual premium of £77. 15*s*. The premium for which an Assurance Office is willing to grant a similar policy for the remaining £4,000 at the time of sale, is £79. 4*s*.; and the cost of an annuity due on the joint life is $12\frac{1}{2}$ years' purchase. It is required to determine the value of the reversion to pay a clear 5 per cent. upon the outlay.

From the value of the estate . . .		£10,000
Deduct the cost of an annuity due, at $12\frac{1}{2}$ years' purchase, of—		
The discount on £10,000 . .	476·19	
,, first premium . . .	77·75	
,, second premium . .	79·20	
	633·14=	7914·25
		£2085·75

It is very questionable, however, whether, under such circumstances, a buyer is entitled to 5 per cent. interest, as the risk of loss is reduced to a minimum. I am inclined to think that 4 per cent. would be a sufficient rate for investments of this kind. The calculation at 4 per cent. would be as under:—

From the value of the estate . . .		£10,000
Deduct the cost of the annuity, viz.—		
Discount at 4 per cent. . .	384·62	
Premium as above . . .	156·95	
	541·57	
By $12\frac{1}{2}$ years' purchase= . .		6769·6
		£3230·4

This is one of a multitude of instances in which policies are not unfrequently offered for sale together with reversionary interests. It would be impossible to give a rule for each case, and nothing but an intimate acquaintance with the mathematical principles of the doctrines of annuities and probabilities will render possible the accurate solution of the many questions that arise in practice.

(*To be continued.*)

On a Plan for making Conditional the Payment of extra Premium in the case of a Life supposed to be Diseased or more than ordinarily Hazardous. By Samuel Younger, *of the British Nation Life Office, and Actuary to the Volunteer Assurance Association.*

The difficulty of dealing equitably with assurers whose prospects of longevity are considered by the medical referee to be below the average of healthy lives of the same age will not, I think, be disputed. The practice of charging a higher premium corresponding to an advanced age, which has hitherto generally been adopted, is, of course, open to the double objection, that the judgment of the medical officer may have been at fault in placing the life in the second or third class at all, and, supposing he were right in his decision thus far, that the premium at which he has assessed the risk may not be in accordance with its precise magnitude. It is true that in after years reductions may be made if the person's health improve; but this plan is not without trouble both to the Office and the assured, and it has the further disadvantage of leaving the latter in a state of uncertainty as to what his future premiums may be. The suggestion of Mr. Morrice Black, that the ordinary premium only might be actually paid on condition that a specified deduction should be made from the sum assured if death happened within a given number of years, seems to be a much more satisfactory arrangement, not only to the assurer—who is thus made acquainted with his real position at the outset, and has, moreover, the chance of escaping entirely from any extra charge—but also to the Company issuing the policy; for it is well known that persons who are taken as second-class lives usually consider themselves unfairly treated, and sometimes refuse, in consequence, to complete the assurance altogether. Now, the system proposed, by converting the extra premium into a reversionary charge, only to be made if the person die within a given period, appears to be a very suitable contrivance for overcoming the difficulty, and cannot be productive of loss to the Office if the amount of the deduction be calculated upon safe principles. My object now is to suggest what appears to be a fair and sufficient amount for this deduction, and to assign the length of the period over which it should extend. The method made use of by Mr. Black, in his investigation of these results, is in accordance with his hypothesis; but as I am inclined to think that other assumptions than those which he employs may be safely

made, by means of which the reversionary deductions in different cases become more consistent in their relative magnitudes, and often considerably lessened in their actual amount, as compared with Mr. Black's figures, I would submit the following points for consideration.

In making an addition to a life the medical officer may, and doubtless does, not unfrequently, commit an error; but, as the mathematical probability of his doing so cannot be determined, our object, in an investigation like the present, must be to introduce hypotheses of such a nature as to counterbalance, to the best of our judgment, the possible error so committed. The first assumption I propose to make is, that the decision of the medical officer, if incorrect, is far more likely to be in favour of the Company than otherwise. There cannot be a doubt that an addition is frequently made to the age when none at all is needed; this must often be the case when the cause assigned is "family history," "hernia," and such like. It cannot *always* follow, for instance, that the life of a man whose father died of consumption is extra-hazardous on that account, for if all the circumstances could be known it might sometimes prove that the disease had its origin long after the birth of the son, and was the consequence of exposure, accident, or unhealthy occupation; yet such a case rarely escapes the charge of some additional premium. The task, however, of discussing in detail the many cases in which an addition is probably made in error, would be endless; but a very great number of instances will readily occur, to those engaged in the business of life assurance, in support of the reasonableness of the assumption that the medical error, when it exists, is far more often in favour of the Company than against it. I would suggest, therefore, that the extra premium proposed to be charged should be left unpaid until the expiration of the period termed the expectation of life, on condition that a certain deduction, agreed upon beforehand, be made from the sum assured if death happen in that period; no deduction to be made if the life fail after that term. In theory, of course, it does not follow that an individual is an average life because he survives the term here spoken of, but it must be remembered we are dealing with a case which theory alone cannot decide in all its bearings, and, under the circumstances, the test seems to be a fair one. It should be distinctly understood that I propose to *assume*, as a *fundamental truth*, that the fact of a person whose life was considered an extra-hazardous one living to the end of the "expectation" period is complete evidence that the medical opinion was wrong, and that no

additional premium was necessary. This being premised, we may reason in the following manner in estimating the amount of the reversionary deduction. If death take place during the first year, the Company should be entitled to deduct one extra premium and a year's interest upon it, to place itself in the same position as if the additional premium had been paid in the ordinary course. If death happen in the second year, the deduction should be two extra premiums, together with interest, and so on. This calculation I carry to the end of the period of the "expectation," and after finding the present values of these several possible deductions, I treat the total as the single premium for an assurance for the whole term of life, and the amount of such assurance I take to be the deduction which should be specified in the policy.

It will be seen that another departure is here made from pure theory, inasmuch as theory would require all the possible deductions to the end of life to be taken into account, instead of those only during the "expectation;" but after very carefully weighing all the circumstances, I am induced to believe that the plan above suggested will give a perfectly safe result for Office use. One consideration alone will go a long way to prove its sufficiency, and that is, that as a large number of perfectly sound lives necessarily fail before the expectation period has passed, therefore, in many instances where an extra premium has been *erroneously* charged, the Office will benefit to the full extent of the deduction by the accidental circumstance of the premature failure of the life. It will be observed that the present value of the several yearly deductions is not treated as the single premium for a *temporary* assurance, to continue only for the number of years expressed by the "expectation," but as the single premium for a whole-life assurance. This is in strict conformity with the assumption already made, that the mere fact of a life surviving the "expectation" period shall of itself be considered as proof that it ought to have been accepted at the ordinary premium. Had the amount of the temporary assurance been taken to represent the deduction, it would have been equivalent to making those persons whose lives at entry were considered doubtful, and who happened to die during the "expectation," pay, not only the extra premiums that had accumulated upon their own policies, but also the accumulations (to the end of the "expectation" period) upon other policies, the owners of which had escaped payment of the same by outliving the specified term. The plan of adopting the amount of a whole-life assurance obviates this contradiction of our fundamental hypothesis, and fixes the

deduction at such a sum as will simply reimburse the Office for the extra premiums left unpaid at death by those whose lives fail to satisfy the proposed test. The one objection that might be raised against the principle generally of converting an extra premium into a reversionary sum—namely, that it will be lost to the Office if the policy lapse, appears to be of very little moment, when the numerous counterbalancing elements are taken into consideration.

In weighing so many different possibilities and probabilities as are presented by the problem before us, a great amount of thought is necessary before any safe conclusion can be arrived at; but I believe, after due reflection, it will be found that the principles here laid down are sufficiently sound to justify the use of the results in practice.

The formula for calculating the deduction may be investigated thus:—taking unity to represent the extra annual premium, the several sums to which the Office would be entitled if death happened in the first, second, or third years, would be $1+i$, $(1+i)+(1+i)^2$, $(1+i)+(1+i)^2+(1+i)^3$, respectively; and if death occurred in the nth year, the amount would be

$$(1+i)\,.\,\frac{(1+i)^n-1}{i}\,.$$

If, therefore, m be the real age of the individual, and e the "expectation of life" at that age, the present value of the several sums will be

$$\Sigma_{\overline{e}|}(p_{m(n-1)}-p_{m(n)})\,.\,\frac{(1+i)^n-1}{i(1+i)^{n-1}}$$

$$=\frac{1+i}{i}\left\{\Sigma_{\overline{e}|}(p_{m(n-1)}-p_{m(n)})-\Sigma_{\overline{e}|}(p_{m(n-1)}-p_{m(n)})\frac{1}{(1+i)^n}\right\}\quad .\quad .\ (1)$$

$$=\frac{1+i}{i}\left\{1-\frac{l_{m+e}}{l_m}-\frac{M_m-M_{m+e}}{D_m}\right\}.$$

This, multiplied by p, the extra premium, and equated to $\frac{M_m}{D_m}\,.\,x$ (x being the equivalent sum payable at death, whenever it may happen) gives

$$x=\left\{\frac{1}{A_m}\left(1-\frac{l_{m+e}}{l_m}\right)-\left(1-\frac{M_{m+e}}{M_m}\right)\right\}.\,\frac{1+i}{i}\times p\quad .\quad .\quad .\ (2)$$

which is, therefore, the amount of the reversionary deduction.

Using the Carlisle table of mortality, and 4 per cent. interest,

we get the following results, which exemplify the formula in a few instances, assuming that the extra premiums made use of correspond with those deduced from the Office tables, when 5, 10, or 15 years are added to the life assured.

Real Age.	Expectation of Life, or period during which the deduction is to be made if death happen.	Medical Addition to the Age, 5 Years.		Medical Addition to the Age, 10 Years.		Medical Addition to the Age, 15 Years.	
		Extra Annual Premium per Cent.	Deduction per Cent. from Sum Assured.	Extra Annual Premium per Cent.	Deduction per Cent. from Sum Assured.	Extra Annual Premium per Cent.	Deduction per Cent. from Sum Assured.
		£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.
30	34 yrs.	0 7 2	6 12 0	0 16 5	15 2 7	1 8 4	26 2 3
40	28 „	0 11 11	8 13 11	1 7 8	20 3 10	2 9 9	36 6 2
50	21 „	1 2 1	10 19 6	2 12 10	26 5 3	4 15 10	47 12 10

The formula (2) expresses the accumulated amount at death of a forborne temporary life annuity (p) for e years. If the summation in (1) were extended to the whole of life, the result would be

$$\frac{1+i}{i}(1-A_m),$$

which is only another form for $1+a_m$; and putting this $=x.A_m$, we have $x=\frac{1+a_m}{A_m}=\frac{1}{\pi_m}$, the common expression for the accumulated amount of a whole life annuity of £1 (payable at the beginning of each year) forborne till death.

Life Assurance in England.

THE change which has come over the condition of this enterprise within the last few years is very remarkable. Only so recently as 1851* we were led to make some observations on its general aspect which were suggested by the state of affairs at that time, and which we now in part repeat with a view to show how curiously that state contrasts with the one at which we have now arrived.

Speaking of the amount of assurance annually effected, we said :—

"Meanwhile, judging from what is passing around, we are disposed to think that the business transacted by the new Companies is to a great extent taken from the old ones, and that the total quantity is very little, if at all, increasing.

* See vol. 2, page 171.

"Not so, however, the aggregate expenditure. Since 1844 certainly not less than fifty new Companies have sprung into existence; and if we estimate the average annual expenses in each of them at £2,000 only, which is probably much below the mark, we have an additional drain upon the pockets of the assured of not less than £100,000 per annum. This, too, it must be remembered, is a wholly unnecessary one; since, were all these new-comers abolished to-morrow, and their policies transferred to the Offices previously established, the latter would have barely sufficient to occupy them. But although any reduction in the amount of business in Companies of the largest calibre has the effect of increasing the rate of expense to the persons assured in them, the loss to the persons so situated is as nothing when compared with that which occurs in a Company lingering on for years with a totally inadequate number of assurances, and terminating at length in a protracted dissolution.

"It is pretty generally known that the rates of premium charged for life assurance are, under ordinary circumstances, about one-third more than sufficient to meet the sum assured; hence it follows, that where the expenses are £3,000 per annum, the premiums must be at least £9,000, to avoid intrenching upon the fund required for the claims. Now, as the expenses almost always rise immediately to their full extent, and most frequently two or three years at least elapse before the premiums come to be upon a par with them merely, it may be imagined in what a condition a Society, going on for a few years only with such a limited amount of business, must eventually find itself; and yet, although numbers have entered upon such a career, and succumbed after a while to the pressure of the surrounding competition, the warning is unheeded, and fresh enterprises are constantly springing up, threatening, like Banquo's issue, to succeed each other till the crack of doom."

The rapid increase in the number of Companies to which these remarks refer, had, as our readers are for the most part aware, been going on for some time, stimulated by the opinion expressed in certain quarters that such increase was needed and that the proper limit would not be attained till every town of importance in the kingdom had its own institution for the assurance of life. In vain were the population returns adduced to prove that there were no means of supporting such a phalanx of Companies. In vain were arguments brought forward to show that institutions of the kind could hope for permanent success only when established on a very large scale: the creation of new ones still continued, till the main occupation of each consisted in a futile struggle to make itself more conspicuous than its neighbours. But the laws which regulate supply and demand are inexorable, and the time has at length arrived when this egregiously mistaken policy has been forced to succumb to them. Within the last eight or nine years no less than 150 Companies have disappeared, and very little observation and consideration will serve to show that many more must follow.

We believe we are quite within the mark in saying that there still remains in this part of the United Kingdom at least six times as many Companies as there are in Scotland, where, as we have seen,* the aggregate income of all of them barely exceeds two millions per annum; and where, as is well known, the competition is so great and the activity in this particular enterprise so unceasing, as to entail a very considerable expenditure. But if the English Companies are to attain to anything like the success and position of the Scottish ones, it is clear that their aggregate income should be six times greater, that is to say, not less than twelve millions per annum—an amount which, after making every allowance for the difference of population, we believe it to be utterly impossible to raise for this purpose. Nor is this the whole case. It must be remembered, that probably one-half of the Scottish income is derived from this part of the United Kingdom, in which but comparatively few Scottish assurances are effected; and hence, to keep pace with our neighbours, the total premium raised here must, in effect, be at least twelve times as great as the amount raised by them, and that too after a very considerable harvest has been reaped by them here. When, therefore, we take into account the vigour and facility with which these institutions establish themselves in Scotland, and the energy and astuteness displayed in their management, we think it will be admitted, that to maintain our numerous establishments on anything like a par with them under such circumstances is out of the question; and since they cannot permanently exist if much below that par, it follows that the number must still be a good deal reduced.

Such being the state of affairs, we may congratulate ourselves that more rational views on this subject are beginning to prevail; and we have especial reason to rejoice that the consequences of our past errors are not more onerous than they are. Happily, the greater number of the extinct Companies have brought their affairs to a close before it was too late, and have thus saved their members from loss and disappointment painful to contemplate. But much of the mischief they have occasioned still remains. Some of the surviving Companies have been forced into expenses which but for the new comers would have been altogether needless, and which have, of course, left the common fund so much less than it otherwise would have been. A practice of incessant and most extensive advertising, too, has been engendered, enormously expensive, out of place, and hardly consistent with the character of

* See page 236 of this Volume.

such institutions; whilst by means of it the public have been led to form ideas altogether extravagant as to the over-sufficiency of their payments and the returns they may expect on account of them—disregarding the simple and obvious truths that the whole premium-income of these establishments is absolutely pledged, with the exception of about 25 per cent. of it; that at least 10 per cent. is commonly absorbed in expenses, and that the remainder, that is to say, 15 per cent of their payments, is, generally speaking, the utmost return they can reasonably look for. The rate of commission has also, we believe, almost universally been raised by the recent competition, and a deduction heavy enough on the ordinary scale made heavier than before. These are some of the inconveniences bequeathed by the extinct Companies to their survivors; but these, it is to be hoped, will in time disappear. If none greater remain, we have good reason to be thankful that the collapse of so huge a bubble has not been attended with worse consequences.—Ed. *A. M.*

*On Triadic Combinations of Fifteen Symbols.** *By* W. S. B. Woolhouse, F.R.A.S., *&c.*

In the *Diary* for 1850 the following mathematical query was proposed by the Rev. T. P. Kirkman, of Croft, near Warrington:—

> "Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast."

Two answers were printed in the *Diary* for 1851; one by the proposer and another by Messrs. Samuel Bills, of Hawton; Thomas Jones, of Chester; Thomas Wainman, of Burley; and W. H. Levy, of Shalborne.

Since that time the problem has acquired considerable notoriety and excited some attention amongst mathematicians. The solutions referred to are irregular in construction, and do not, in their present state, suggest any system of derivation. A complete and systematic solution, with a few observations, may, therefore, be acceptable to some of the readers of the *Diary*.

The number 7 can be partitioned in three unequal portions

* We reprint this as the work of a gentleman eminent for his actuarial attainments. —Ed. *A. M.*

only by the components 1, 2, 4; and taking the numbers 1, 2, as first differences, we deduce two triads, viz., 1 2 4, 1 3 4. By augmenting each of these successively by unity, observing to return cyclically from 7 to 1, we get two sets of seven triads out of seven things without repeating a pair, viz.:—

124, 235, 346, 457, 561, 672, 713;
134, 245, 356, 467, 571, 612, 723.

By taking the complements of 7, the triad 1 2 4 gives 6 5 3 or 3 5 6, which belongs to the second set.

If the fifteen young ladies will excuse our designating them by the symbols h; 1, (1); 2, (2); . . . . 7, (7), it will be evident from the materials here stated that a solution of the problem will be at once obtained, in like manner, by operating cyclically upon the numerals of any of the six following primary combinations:—

1.	2.	3.	4.	5.	6.
h 7 (7)	h 7 (7)	h 7 (7)	h 7 (7)	h 7 (7)	h 7 (7)
1 2 4	1 2 4	1 2 4	3 5 6	3 5 6	3 5 6
3 (2) (5)	3 (4) (6)	3 (1) (5)	4 (2) (5)	1 (2) (6)	4 (2) (6)
5 (1) (6)	5 (2) (3)	5 (4) (6)	2 (1) (6)	2 (4) (5)	1 (4) (5)
6 (3) (4)	6 (1) (5)	6 (2) (3)	1 (3) (4)	4 (1) (3)	2 (1) (3)

After developing each of these, it will be found that Nos. 4, 5, 6, give the same combinations as 1, 2, 3, only in a retrograde order. Every solution of the problem must, therefore, be comprised in the systems 1, 2, and 3; and those given in the *Diary* will be found to belong to No. 1.

For the purpose of illustration it will be sufficient to put down the following development of No. 1:—

1.	2.	3.	4.	5.	6.	7.
h 7 (7)	h 1 (1)	h 2 (2)	h 3 (3)	h 4 (4)	h 5 (5)	h 6 (6)
1 2 4	2 3 5	3 4 6	4 5 7	5 6 1	6 7 2	7 1 3
3 (2) (5)	4 (3) (6)	5 (4) (7)	6 (5) (1)	7 (6) (2)	1 (7) (3)	2 (1) (4)
5 (1) (6)	6 (2) (7)	7 (3) (1)	1 (4) (2)	2 (5) (3)	3 (6) (4)	4 (7) (5)
6 (3) (4)	7 (4) (5)	1 (5) (6)	2 (6) (7)	3 (7) (1)	4 (1) (2)	5 (2) (3)

Here it will be observed that each of the seven combinations is derivable from that which precedes it by a fixed law of succession. This law can be modified by means of permutation, which is only equivalent to substitution; thus, for example—

h	7	(7)	h	1	(1)	
2	1	4	3	2	5	
(2)	3	(5)	(3)	4	(6)	&c.
(6)	(1)	5	(7)	(2)	6	
6	(3)	(4)	7	(4)	(5)	

which we shall adopt as leading to more symmetrical relations.

By simply changing the notation, the three systems may be exhibited in full as follows:—

	1.	2.	3.	4.	5.	6.	7.
No. 1	$a_1b_1c_1$ $a_2b_2c_2$ $a_3b_3c_3$ $a_4b_4c_4$ $a_5b_5c_5$	$a_1b_2b_4$ $b_3a_2c_4$ $b_5c_2a_4$ $c_1a_3a_5$ $b_1c_5c_3$	$a_1a_2a_3$ $c_2b_3a_5$ $c_5c_4c_1$ $b_4b_5b_1$ $b_2c_3a_4$	$a_1b_3b_5$ $c_4c_2b_1$ $c_3a_5b_4$ $a_3c_5b_2$ $a_2a_4c_1$	$a_1c_2c_5$ $a_5c_4b_2$ $a_4b_1a_3$ $b_5c_3a_2$ $b_3c_1b_4$	$a_1c_4c_3$ $b_1a_5a_2$ $c_1b_2b_5$ $c_5a_4b_3$ $c_2b_4a_3$	$a_1a_5a_4$ $b_2b_1b_3$ $b_4a_2c_5$ $c_3c_1c_2$ $c_4a_3b_5$
No. 2	$b_1a_1c_1$ $a_2c_2b_2$ $a_3b_3c_3$ $b_4c_4a_4$ $c_5a_5b_5$	$b_1a_2a_5$ $c_2a_3b_4$ $b_2b_5c_1$ $c_5a_4b_3$ $a_1c_4c_3$	$b_1c_2c_4$ $a_3b_2c_5$ $b_4c_3a_5$ $a_1b_3b_5$ $a_2a_4c_1$	$b_1a_3a_4$ $b_2b_4a_1$ $c_5c_1c_4$ $a_2b_5c_3$ $c_2b_3a_5$	$b_1b_2b_3$ $b_4c_5a_2$ $a_1a_5a_4$ $c_2c_3c_1$ $a_3b_5c_4$	$b_1b_4b_5$ $c_5a_1c_2$ $a_2c_4b_3$ $a_3c_1a_5$ $b_2c_3a_4$	$b_1c_5c_3$ $a_1a_2a_3$ $c_2a_4b_5$ $b_2a_5c_4$ $b_4c_1b_3$
No. 3	$a_1b_1c_1$ $a_2b_2c_2$ $a_3b_3c_3$ $a_4b_4c_4$ $a_5b_5c_5$	$a_1a_2b_3$ $b_2a_3a_4$ $c_2b_5c_4$ $a_5c_3c_1$ $b_1c_5b_4$	$a_1b_2b_5$ $a_3c_2a_5$ $a_4c_5c_1$ $b_1c_4b_3$ $a_2b_4c_3$	$a_1a_3c_5$ $c_2a_4b_1$ $a_5b_4b_3$ $a_2c_1b_5$ $b_2c_3c_4$	$a_1c_2b_4$ $a_4a_5a_2$ $b_1c_3b_5$ $b_2b_3c_5$ $a_3c_4c_1$	$a_1a_4c_3$ $a_3b_1b_2$ $a_2c_4c_5$ $a_3b_5b_4$ $c_2c_1b_3$	$a_1a_5c_4$ $b_1a_2a_3$ $b_2c_1b_4$ $c_2c_5c_3$ $a_4b_3b_5$

These three systems are independent of each other, and cannot be mutually elicited by any kind of substitution. The primary combination of No. 2 has been chosen to show the remarkable fact that the first and second systems can be made up of the same set of thirty-five triads; and, collecting these triads, they may be arranged as follows:—

$$a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3 + a_4b_4c_4 + a_5b_5c_5$$

$$\begin{array}{l|l|l|l|l|l|l} +a_1 & a_2a_3+b_1 & b_2b_3+c_1 & c_2c_3+a_2 & b_3c_4+b_2 & c_3a_4+c_2 & a_3b_4 \\ & a_4a_5 & b_4b_5 & c_4c_5 & b_4c_5 & c_4a_5 & a_4b_5 \\ & b_2b_4 & c_2c_4 & a_2a_4 & b_5c_3 & c_5a_3 & a_5b_3 \\ & b_3b_5 & c_3c_5 & a_3a_5 & & & \\ & c_2c_5 & a_2a_5 & \multicolumn{4}{l}{b_2b_5 + a_5b_4c_3 + b_5c_4a_3 + c_5a_4b_3.} \\ & c_3c_4 & a_3a_4 & b_3b_4 & & & \end{array}$$

The symmetrical association amongst this set of triads is apparent, and if given promiscuously it would not be difficult to arrange them.

It is also evident that, as regards No. 1, the set of triads will remain unaltered if we cyclically permute the three vertical columns of the primary combination.

The triads contained in the third system are essentially unsymmetrical, and admit of only one systematic distribution.

The three systems, numerically expressed, may be stated as follows :—

First System.

1.	2.	3.	4.	5.	6.	7.
1 2 3	1 5 11	1 4 7	1 8 14	1 6 15	1 12 9	1 13 10
4 5 6	8 4 12	6 8 13	12 6 2	13 12 5	2 13 4	5 2 8
7 8 9	14 6 10	15 12 3	9 13 11	10 2 7	3 5 14	11 4 15
10 11 12	3 7 13	11 14 2	7 15 5	14 9 4	15 10 8	9 3 6
13 14 15	2 15 9	5 9 10	4 10 3	8 3 11	6 11 7	12 7 14

Second System.

1.	2.	3.	4.	5.	6.	7.
2 1 3	2 4 13	2 6 12	2 7 10	2 5 8	2 11 14	2 15 9
4 6 5	6 7 11	7 5 15	5 11 1	11 15 4	15 1 6	1 4 7
7 8 9	5 14 3	11 9 13	15 3 12	1 13 10	4 12 8	6 10 14
11 12 10	15 10 8	1 8 14	4 14 9	6 9 3	7 3 13	5 13 12
15 13 14	1 12 9	4 10 3	6 8 13	7 14 12	5 9 10	11 3 8

Third System.

1.	2.	3.	4.	5.	6.	7.
1 2 3	1 4 8	1 5 14	1 7 15	1 6 11	1 10 9	1 13 12
4 5 6	5 7 10	7 6 13	6 10 2	10 13 4	13 2 5	2 4 7
7 8 9	6 14 12	10 15 3	13 11 8	2 9 14	4 12 15	5 3 11
10 11 12	13 9 3	2 12 8	4 3 14	5 8 15	7 14 11	6 15 9
13 14 15	2 15 11	4 11 9	5 9 12	7 12 3	6 3 8	10 8 14

By collating each succeeding combination *seriatim* with the primitive arrangement, and noting the order of the particular line from which each symbol is derived, the following tables are readily formed :—

Order of Derivation.—First System.

2.	3.	4.	5.	6.	7.
124	123	135	125	143	154
324	235	421*	542	152*	213*
524	541*	354	413*	125*	425
135*	451*	352	532	543	312*
153*	234	241*	314*	243	435

Order of Derivation.—Second System.

2.	3.	4.	5.	6.	7.
125*	124*	134	123*	145	153
234	325	241	452	512	123
251*	435	514	154	243*	245*
543	135	253*	231*	315	254*
143	241*	235*	354	234*	413

Order of Derivation.—Third System.

2.	3.	4.	5.	6.	7.
123	125	135	124	143	154
234	325	241	452	512	123
254	451	543	135	245	214
531	143	215	235	354	253
154	243	234	341	213	435

By means of these tables alone a system of combinations can be constructed with facility from any primary by taking the symbols *ad libitum* from the lines indicated by the corresponding numbers. The stars are placed to show where the three numbers appear in duplicate.

To systematize any given solution and ascertain to which system it belongs, take any one of the given daily arrangements as the primary, then collating the others with it, write out a table like the preceding. The duplicates found in this table will contain a certain number in common, which will indicate the particular line of the primary that must be placed first in the first system or second in the second system. In each column of the table the same number will occur once more; if on examination these be found all to refer to the same identical symbol, that will be the leading symbol of the first system; but if they be found to refer to the three symbols of the triad, the solution will belong to the second system, and must be treated accordingly.

If no duplicates appear in the table, this fact will at once characterize the third system.

The triads of No. 1 and No. 2 can be formed into a system in a great variety of ways; out of the 35 triads can be constructed 56 synthetic arrangements, each containing the 15 symbols, and these can again be all exhibited in 8 distributions. Adopting numerals for simplicity, these 8 distributions may be put down as follows :—

First System.

	1.	2.	3.	4.	5.	6.	7.
No. 1	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	1 5 11 8 4 12 14 6 10 3 7 13 2 15 9	1 4 7 6 8 13 15 12 3 11 14 2 5 9 10	1 8 14 12 6 2 9 13 11 7 15 5 4 10 3	1 6 15 13 12 5 10 2 7 14 9 4 8 3 11	1 12 9 2 13 4 3 5 14 15 10 8 6 11 7	1 13 10 5 2 8 11 4 15 9 3 6 12 7 14
No. 2	1 2 3 4 6 5 7 12 14 10 15 8 13 9 11	1 6 15 12 4 8 9 5 10 3 7 13 2 11 14	1 4 7 5 12 13 11 8 3 15 9 2 6 14 10	1 12 9 8 5 2 14 13 15 7 11 6 4 10 3	1 5 11 13 8 6 10 2 7 9 14 4 12 3 15	1 8 14 2 13 4 3 6 9 11 10 12 5 15 7	1 13 10 6 2 12 15 4 11 14 3 5 8 7 9
No. 3	1 2 3 7 9 8 4 15 11 13 12 5 10 6 14	1 9 12 15 7 5 6 8 13 3 4 10 2 14 11	1 7 4 8 15 10 14 5 3 12 6 2 9 11 13	1 15 6 5 8 2 11 10 12 4 14 9 7 13 3	1 8 14 10 5 9 13 2 4 6 11 7 15 3 12	1 5 11 2 10 7 3 9 6 14 13 15 8 12 4	1 10 13 9 2 15 12 7 14 11 3 8 5 4 6
No. 4	1 2 3 10 12 11 13 6 8 4 9 14 7 15 5	1 12 9 6 10 14 15 11 4 3 13 7 2 5 8	1 10 13 11 6 7 5 14 3 9 15 2 12 8 4	1 6 15 14 11 2 8 7 9 13 5 12 10 4 3	1 11 5 7 14 12 4 2 13 15 8 10 6 3 9	1 14 8 2 7 10 3 12 15 5 4 6 11 9 13	1 7 4 12 2 6 9 10 5 8 3 11 14 13 15
No. 5	1 2 3 13 15 14 10 9 5 7 6 11 4 12 8	1 15 6 9 13 11 12 14 7 3 10 4 2 8 5	1 13 10 14 9 4 8 11 3 6 12 2 15 5 7	1 9 12 11 14 2 5 4 6 10 8 15 13 7 3	1 14 8 4 11 15 7 2 10 12 5 13 9 3 6	1 11 5 2 4 13 3 15 12 8 7 9 14 6 10	1 4 7 15 2 9 6 13 8 5 3 14 11 10 12
No. 6	1 2 3 7 14 12 4 11 15 13 8 6 10 5 9	1 14 8 11 7 6 5 12 13 3 4 10 2 9 15	1 7 4 12 11 10 9 6 3 8 5 2 14 15 13	1 11 5 6 12 2 15 10 8 4 9 14 7 13 3	1 12 9 10 6 14 13 2 4 5 15 7 11 3 8	1 6 15 2 10 7 3 14 5 9 13 11 12 8 4	1 10 13 14 2 11 8 7 9 15 3 12 6 4 5
No. 7	1 2 3 10 8 15 13 5 12 4 14 9 7 11 6	1 8 14 5 10 9 11 15 4 3 13 7 2 6 12	1 10 13 15 5 7 6 9 3 14 11 2 8 12 4	1 5 11 9 15 2 12 7 14 13 6 8 10 4 3	1 15 6 7 9 8 4 2 13 11 12 10 5 3 14	1 9 12 2 7 10 3 8 11 6 4 5 15 14 13	1 7 4 8 2 5 14 10 6 12 3 15 9 13 11
No. 8	1 2 3 13 11 9 10 14 6 7 5 15 4 8 12	1 11 5 14 13 15 8 9 7 3 10 4 2 12 6	1 13 10 9 14 4 12 15 3 5 8 2 11 6 7	1 14 8 15 9 2 6 4 5 10 12 11 13 7 3	1 9 12 4 15 11 7 2 10 8 6 13 14 3 5	1 15 6 2 4 13 3 11 8 12 7 14 9 5 10	1 4 7 11 2 14 5 13 12 6 3 9 15 10 8

It will be observed that the number 1 is the leading symbol in all these assortments. By taking any other number as the leading symbol, another class of eight assortments can be constructed in like manner, embodying the same set of 56 synthetic arrangements, but all of them differently distributed. In this way 120 assortments, or solutions, according to the first system, are obtained

from the same set of triads. Moreover, from every one of these eight different solutions can be constructed on the second system; so that, from the same set of thirty-five triads, it is possible to construct no less than 1,080 different solutions of the problem.

The initial arrangements of the first system can be chosen under so many different forms, that any solution to the problem arrived at by a tentative process is almost sure to belong to this system, and is unlikely to belong to the third system, which is so restricted as not to admit of modification. The solutions given in the *Diary* for 1851 both belong to the first system. To make this apparent we here annex them, systematically arranged by the process pointed out.

1.	2.	3.	4.	5.	6.	7.
$a_3a_2a_1$	$a_3b_3c_3$	$a_3b_1d_1$	$a_3d_3e_3$	$a_3b_2e_2$	$a_3c_2d_2$	$a_3e_1c_1$
$b_1b_3b_2$	$d_3b_1c_2$	$b_2d_3e_1$	$c_2b_2a_2$	$e_1c_2b_3$	$a_2e_1b_1$	$b_3a_2d_3$
$d_1d_3d_2$	$e_3b_2c_1$	$e_2c_2a_1$	$d_2e_1c_3$	$c_1a_2d_1$	$a_1b_3e_3$	$c_3b_1e_2$
$c_1c_3c_2$	$a_1d_1e_1$	$c_3e_3a_2$	$d_1e_2b_3$	$e_3d_2b_1$	$e_2c_1d_3$	$d_2a_1b_2$
$e_1e_3e_2$	$a_2e_2d_2$	$b_3d_2c_1$	$b_1c_1a_1$	$d_3a_1c_3$	$b_2c_3d_1$	$c_2d_1e_3$
5 14 11	5 10 15	5 13 8	5 4 1	5 7 2	5 9 12	5 3 6
13 10 7	4 13 9	7 4 3	9 7 14	3 9 10	14 3 13	10 14 4
8 4 12	1 7 6	2 9 11	12 3 15	6 14 8	11 10 1	15 13 2
6 15 9	11 8 3	15 1 14	8 2 10	1 12 13	2 6 4	12 11 7
3 1 2	14 2 12	10 12 6	13 6 11	4 11 15	7 15 8	9 8 1

NOTES AND QUERIES.

A Query about Interest Accounts. *By* PROFESSOR DE MORGAN.

THE following incident, and other things not quite so remarkable, lead me to think that many interest accounts may possibly be erroneously kept: at any rate it will appear that a query is not out of place.

Many years ago, one friend of mine borrowed of another: both have been dead a long time. The borrower was a competent mathematician, a practised accountant, and in knowledge an actuary. The debtor gave his note of hand, and a policy of assurance—that is, he agreed to pay the ordinary premium to the creditor, that the creditor might insure the debtor's life for himself. The creditor chose to become his own insurer, and put the premiums in his pocket—it being understood, of course, that he would have no claim on the debtor's executor. Many years passed over, during which the interest and premiums were irregularly paid. At last the debtor desired to balance the account, and it was agreed

that the premiums should be taken to have been instalments of principal, by which, of course, it would follow that a portion of what had been called interest would be reckoned as further payment of principal. The debtor was requested to make out his account of the sum remaining due from him, and the creditor requested me to examine it. Let it be understood that the propriety of the arrangement had nothing to do with the difficulty which arose; the parties had agreed to it, and I was to assume it. The debtor presented his account, by which it appeared that *he had overpaid*, and had a balance to claim. I felt pretty sure that this could not be, and I accordingly made out an account on the straightforward plan of adding the interest to principal for each interval, deducting the whole of each payment as it was made, and computing interest on the remainder until the next payment. As each payment was more than the interest then due, it is clear that there was no compound interest introduced by this plan. The result was that the balance was against the debtor; and I wrote to inform him that his account was erroneous, and that I should be glad to see him on the subject. After some days he called upon me about another matter, and I asked him if he had considered my note. "Not a bit of it," said he; "I took it for granted you would have found out your mistake before this." "There is no mistake on my part," said I. "I'm glad you think so," said he; "I see I shall have the honour of teaching you something for once in my life." I replied that I would make *his* error as clear as a proposition in Euclid. I then put this question:—"A person borrows £100 on the 1st of January, at 5 per cent. payable yearly: on each 1st of January, for three years then ensuing, he pays £5. Immediately after the third payment of interest he desires to settle the account; what is he to pay?" My friend considered the matter for about a minute, evidently in the spirit of old Professor Vince, who would not grant an opponent that the whole was greater than its part until he saw what use was to be made of it. At last he said, "I suppose he must repay the £100." I replied, "Take time, for I shall not let you retract. Are you clear that a debtor who pays his interest to the day can balance the account by repayment of principal the moment after any payment of interest?" When this was fully admitted, I said—"Now make out the account I have just supposed in your own way." He did so, and after some minutes he raised his head and said—"It comes out that he has more to pay!" The matter was then soon cleared up, and the true balance was paid. Had it ended here I should have supposed my friend had a way of his own; but not long afterwards he informed me that I was wrong after all, for that he had sent an interest account on an amended plan to a solicitor, who had returned it as wrong, and insisted upon the plan which he had always followed until I set him right. It is so long ago that I can hardly venture to state his method; but, all things considered, I think it may be useful to suggest inquiry into the usual plan, by which some clear statement may be brought out. For some time afterwards I looked narrowly at the accounts which came in my way, but they were all *bankers'* accounts, which I found correct. That an interest account may more easily be wrong than any other will be readily granted.

CORRESPONDENCE.

THE LAW OF HUMAN MORTALITY.

To the Editor of the Assurance Magazine.

Sir,—The importance of the discovery of the law of human mortality is too obvious to require any further explanation, and it is highly gratifying to observe that the labours which have been brought to bear upon the discovery of these laws, and which for a time seemed to have ceased, have lately been renewed. The reader of the *Assurance Magazine* must feel indebted to this periodical for the publication of any essay referring to this question, for even a fruitless endeavour in this direction may be the means of leading another inquirer into the right path; and from this point of view even the contest which has sprung up between Mr. Gompertz and Mr. Edmonds, although not pleasing in itself, may prove of advantage to the question in point.

I have not the least intention of questioning the value of Mr. Edmonds's exertions, and I neither can nor will interfere in the contest as to the priority of the discovery; but when Mr. Edmonds opposes principally Mr. Gompertz on the ground of his formula giving only a numerical approximation, while he asserts to have found out a law of nature—when he places the result of his labours beyond all doubt—when he does not hesitate to assert (*Assurance Magazine*, ix., p. 177), "The truth is, however, that (p), with its three determinate values, is independent of all formulæ, has existed as long as man has existed, and forms part of the foundations of the universe," then I venture to object, begging to add a few remarks. I fear that such assertions as those above mentioned may easily lead the student in an entirely false direction. I think the only point we can arrive at in the present state of science is to discover a formula which gives a numerical approximation to the numbers contained in the table of mortality; any discovery of the natural law of mortality cannot be contained in a formula only—on the contrary, thereby the elements which constitute the formula must be explained in their reference to the effect of death.

I beg permission to add a few explanatory remarks.

When we derive any conclusions from a table of mortality, we thereby pronounce our conviction that the table contains the expression of a natural law, although it may be only a numerical approximation to this law. If we assume (according to Deparcieux), as a general rule, that, out of 814 persons aged 20 years, after the lapse of a year only 806 are alive, this will say neither more nor less than that certain for the present unknown powers, acting in persons aged 20 years in accordance with certain for the present unknown laws, undergo such a change in the course of a year, that the one side of their effect, which we call "life," is thus changed, that while its measure has been expressed by 814 in the beginning of the year, it is only 806 at the end of it, supposing we considered a sufficient number of cases, so that the disturbances from other unknown accidental causes may be left out of consideration. I do not see the least reason to suppose that the phenomenon which we call "life" is the effect of a single cause, but just as little reason to suppose that this phenomenon be the only effect of the force or forces which are its cause. We will call the sum of the

single forces (resulting force) which are the cause of "life," however, only with reference to this utterance of their activity and with the exclusion of those parts acting in other directions, the "power of life," without troubling ourselves as to this being a single or a composed force, this question being immaterial as to the numerical results. It is obvious that even if we be so fortunate as to discover the law of the changes of this power, we cannot draw any conclusions as to its components, nothing at all being known to us of their nature.

But this becomes still more clear by another peculiarity, which should always be borne in mind. As long as we consider the number given by the table of mortality in its totality as a unit, the "power of life" offers many analogies with other natural forces. It is a changeable force, which can be made known and fixed by its changes, and its intensity is measured by the numbers of the persons alive. But the thing suddenly changes when we divide our field of observation in its single parts—the individuals. Of the 814 persons alive a year ago, 8 are dead and 806 alive; in each individual the "power of life," in the sense in which we have defined this expression, is the same as it has been a year before, and in the 8 persons dead it is entirely extinct. While we have observed a decrease of the power of life in the total number considered as a unit, two states opposed to each other appear in the individuals; they must be considered as each having always the same intensity, as there cannot be a question of a gradual passing from the one state to the other in the sense in which we mean life and death. Therefore, in the individual that expression of the force which we have defined as "power of life" cannot be observed at all; and on closer examination we perceive that always two elements enter in the measure of the "power of life," which, like all forces, can only be measured by its effects. One of these elements consists of the number alive at an earlier age, or, if we prefer it, of the number deceased since a certain time. It follows that the "power of life" can only be measured by a quotient, and is considered proportionate to the number alive. If we try to make this clear, we may assume that the power of life existing in all persons born at a certain period gives a sum which, divided by the number born, has as quotient a; this force is equally distributed over the whole number, leaving to each a quantity equal to a, and this quantity be at the same time necessary and sufficient to permit of that state which we call life. In the time following the whole quantity of the power of life diminishes, but the remaining part contracts itself at the same time, so that the intensity in the individual remains equal to a, and, all fractional values left out of consideration, the remaining quantity of the power of life can no longer be expanded over the whole number of persons, so that some of them contain no power of life at all—*i.e.*, they die—while those alive must be considered to be so always by the same amount of this force from birth to death. If we say that the mortality is greater at 60 than at 20, this must not be understood as affirming that the power of life has a greater intensity in an individual aged 20 than in one aged 60, and for the same reasons we cannot imply an increase of the power of life when we say that the mortality decreases from birth to the age of 10 years. On the contrary, these expressions have reference to quite another quantity, they refer to the proportion of two numbers of persons alive at different equidistant ages, or, what is the same, to the proportion of the power of life at equal intervals of age—*i.e.*, the probability to live a certain time.

If we denote the probability that a person aged x will live the interval of time Δ, by $\frac{L_{(x+\Delta)}}{L_{(x)}}$, this is evidently equal to

$$\frac{L_{\left(x+\frac{\Delta}{n}\right)}}{L_{(x)}} \cdot \frac{L_{\left(x+\frac{2\Delta}{n}\right)}}{L_{\left(x+\frac{\Delta}{n}\right)}} \cdot \frac{L_{\left(x+\frac{3\Delta}{n}\right)}}{L_{\left(x+\frac{2\Delta}{n}\right)}} \cdot \cdot \cdot \cdot \frac{L_{(x+\Delta)}}{L_{\left(x+\frac{(n-1)\Delta}{n}\right)}};$$

therefore,

$$\log \frac{L_{(x+\Delta)}}{L_{(x)}} = \log\left[\frac{L_{\left(x+\frac{\Delta}{n}\right)}}{L_{(x)}}\right] + \log\left[\frac{L_{\left(x+\frac{2\Delta}{n}\right)}}{L_{\left(x+\frac{\Delta}{n}\right)}}\right] + \log\left[\frac{L_{\left(x+\frac{3\Delta}{n}\right)}}{L_{\left(x+\frac{2\Delta}{n}\right)}}\right] + \log \cdot \cdot \cdot \cdot$$

and, if we take n infinitely large,

$$\log \frac{L_{(x+\Delta)}}{L_{(x)}} = \int_{x}^{x+\Delta} \frac{d.L_{(x)}}{L_{(x)}}.$$

Now $\frac{d\,L_{(x)}}{L_{(x)}}$ is nothing else but the probability to die in the next moment taken inversely, $1 - \frac{L_{(x+dx)}}{L_{(x)}} = \frac{L_{(x)} - L_{(x+dx)}}{L_{(x)}} = \frac{-d L_{(x)}}{L_{(x)}}$, while the logarithm of the probability to live still in the next moment is $d.L_{(x)}$. These probabilities form, either consciously or unconsciously, the first step in all our considerations as to mortality—we refer to them when we speak of an increase or a decrease in the power of life—and in reality they contain all changes of this power, although they do not directly express them.

If we denote the probability to die in the next moment by the derived function $\phi'_{(x)}dx$, the number of persons dying out of $L_{(x)}$ in the next moment must be $L_{(x)}\phi'_{(x)}dx$; and as the change is a decrease, $\frac{dL_{(x)}}{L_{(x)}} = -\phi'_{(x)}dx$, $\log k.L_{(x)} = -\phi_{(x)}$, $k.L_{(x)} = e^{-\phi_{(x)}}$, k being the constant of integration. It is obvious that if it is possible to express the law of mortality by a complete function of the age x (with the only exception of the logarithmic function), the analytical form of $L_{(x)}$ will be $e^{-\phi_{(x)}}$, while all other forms of $L_{(x)}$ would express that the mortality either must be a logarithmic function, or cannot be given in a complete form at all; and thus we are induced, in our examinations of the numbers alive, to use the form $e^{-\phi_{(x)}}$, which considerations lead us to choose the logarithms instead of the numbers in our investigations as to the nature of $\phi_{(x)}$.

The difference of two of these logarithms, $\log L_{(x+\Delta)} - \log L_{(x)}$, is, as we have seen, the integral of $\phi'_{(x)}dx$ in the limits x and $x+\Delta$ taken inversely; and in this value, better in $\phi'_{(x)}$, we must look for the qualities so well characterized by Mr. Edmonds; but it ought to be borne in mind that we cannot expect to find anything but numerical results and numerical laws.

I must beg your indulgence for having trespassed so largely on your space,

And remain,
Sir,
Yours most obediently,
WILHELM LAZARUS.

Hamburg, 15th August, 1861.*

* We are enabled at length to find room for this letter, referred to in our "Notice to Correspondents," October, 1861.—Ed. *A. M.*

MR. PETER GRAY'S DEMONSTRATION OF FORMULÆ.

To the Editor of the Assurance Magazine.

Sir,—The demonstration of the expressions for the values of single and annual premiums given by Mr. P. Gray (No. XLVIII., page 238, of the *Assurance Magazine*) is quite new and remarkable indeed, but yet it affords us a greater interest, if we suppose, for the sake of generalization, the consideration for forbearance *in infinitum*, we have

$$A_x=(1-v)a'_x(A_x+A^2_x+A^3_x\ldots\ldots) \quad . \quad . \quad . \quad \text{(I.)}$$

which, as is known, is equal to

$$(1-v)a'_x\cdot\frac{A_x}{1-A_x},$$

and thus the expression for the present value will become

$$A_x=1-(1-v)a'_x.$$

Hence, dividing the foregoing formula (I.), by a'_x, we obtain immediately the annual premium

$$\pi_x=(1-v)(A_x+A^2_x+A^3_x+\ \ldots),$$

which is equal to the expression

$$\pi_x=\frac{(1-v)A_x}{1-A_x}.$$

I have the honour to be,
Sir,
Your most obedient servant,

D. AUGUST WIEGARD,
Director of Life Assurance Society "Iduna."

Halle a/S. Prussia, Germany,
23rd August, 1862.

ON INCREASING AND DECREASING SCALES OF PREMIUM.

To the Editor of the Assurance Magazine.

Sir,—The following lines may have an interest for some of the junior members of the Institute, and that is the only reason for my venturing to address you upon a subject so simple as that of determining a premium, ascending or descending by a series of stages, for a whole-term life assurance.

In practice I have met with five varieties of this form of payment, viz.:—

1st. The premium to be p for the first stage, and to be increased or decreased so as to be $p\pm q$ (q being a quantity previously determined) for the second stage, $p\pm 2q$ for the third stage . . . and $p\pm(v-1)q$ for the vth stage, at which it is to remain constant for the remainder of life; to find the value of p.

2nd. The premium for the first stage to be p (p being determined), for the second stage $p\pm q$ (q being arbitrarily fixed), for the third stage $p\pm 2q$, &c., and for the vth stage $p\pm q'$, at which it is to remain constant; to find the value of q'.

3rd. The premium for the first stage being determined, to find the increase or decrease to be paid at each of the subsequent stages.

4th. The premium for the first, second, third, &c., stages being fixed at p, p_1, p_2, &c., to find the premium to be paid for the vth stage, p_v, at which it is to remain constant.

5th. The premium for the first stage to be p, and to be increased or decreased by a proportion of p, so as to be $p+\frac{1}{q}p=p\left(1+\frac{1}{q}\right)$ for the second stage, $p\left(1+\frac{2}{q}\right)$ for the third stage . . . $p\left(1+\frac{v-1}{q}\right)$ for the vth stage, at which it is to remain constant.

These cases may be more shortly stated thus:—

1st. q being known, to find p.
2nd. p and q being known, to find q'.
3rd. p being known, to find q.
4th. p, p_1, p_2 . . . &c., being known, to find the increase to or decrease from p_{v-1} to form p_v.
5th. p to be increased or decreased at each series by $\frac{p}{q}$.

I. An example of the first of these forms will be found in *Jones on Annuities, &c.*, p. 194, viz.:—

"Suppose the annual premium to increase or decrease a *certain* sum every t years, and, at the end of v intervals of t years each, the premium to continue constant during the remainder of life; what annual premium should be required during the *first* t years?

"Let p=the annual premium required;
q=the increase or decrease *per* £1* every t years."

Departing for a moment from the text, the argument for this case may be stated as follows:—

Benefit term.

An assurance of £1 at death, the present value of which	$=A_x$.

Payment terms.

An annuity of £p for the whole term of life, the present value of which	$=p(1+a_x)$.
± A deferred annuity of q (q being arbitrarily determined), first payment or deduction at end of t years	$=q(a_{(x)}\rceil_{t-1})$.
± A further deferred annuity of q; first payment or deduction at end of $2t$ years	$=q(a_{(x)}\rceil_{2t-1})$.
. . . . .	
± A further deferred annuity of q; first payment or deduction at end of vt years	$=q(a_{(x)}\rceil_{vt-1})$.

Equating, we have, as stated by Jones (*ibid.*):—

"$A_x=p(1+a_x)\pm q(a_{(x)}\rceil_{t-1}+a_{(x)}\rceil_{2t-1}+a_{(x)}\rceil_{3t-1}+\ldots+a_{(x)}\rceil_{vt-1})$;

* This, taken with the context, is evidently intended for the sum assured, not per £1 of the premium.

by transposition and division,

$$p=\frac{\mathrm{A}_x\mp q(a_{(x)}\overline{\rceil}_{t-1}+a_{(x)}\overline{\rceil}_{2t-1}+a_x\overline{\rceil}_{3t-1}+\ldots+a_x\overline{\rceil}_{vt-1})}{1+a_x};$$

by substitution in the first equation,

$$\frac{\mathrm{M}_x}{\mathrm{D}_x}=p.\frac{\mathrm{N}_{x-1}}{\mathrm{D}_x}\pm q\left(\frac{\mathrm{N}_{x+t-1}+\mathrm{N}_{x+2t-1}+\mathrm{N}_{x+3t-1}+\ldots+\mathrm{N}_{x+vt-1}}{\mathrm{D}_x}\right);$$

from which we obtain

$$p=\frac{\mathrm{M}_x\mp q(\mathrm{N}_{x+t-1}+\mathrm{N}_{x+2t-1}+\mathrm{N}_{x+3t-1}+\ldots+\mathrm{N}_{x+vt-1}}{\mathrm{N}_{x-1}}."$$

A numerical example follows at Carlisle 4 per cent., but for convenience I substitute one at Experience 3 per cent., viz.:—

Example.—"What annual premium should be required during the first 5 years to insure £100 on a life aged 31, the annual premium to increase 4*s*. every 5 years, and remain constant at the end of 20 years (Experience 3 per cent.)."

$$t=5, \qquad v=4, \qquad q=\frac{\cdot 2}{100}=\cdot 002.$$

$$\begin{array}{rr}
\mathrm{N}_{x+t-1}=\mathrm{N}_{35}= & 543540\cdot 812\\
\mathrm{N}_{x+2t-1}=\mathrm{N}_{40}= & 412864\cdot 847\\
\mathrm{N}_{x+3t-1}=\mathrm{N}_{45}= & 305877\cdot 513\\
\mathrm{N}_{x+4t-1}=\mathrm{N}_{50}= & 219141\cdot 537\\
\hline
 & 1481424\cdot 709\\
 & 200\\
\hline
 & 2962\cdot 849\\
\mathrm{M}_x=\mathrm{M}_{31}= & 13770\cdot 543\\
\hline
 & 10807\cdot 694
\end{array}$$

$$\mathrm{N}_{x-1}=\mathrm{N}_{30}=\frac{10807\cdot 694}{702266\cdot 223}=\cdot 01539\times 100=£1.\ 10s.\ 10d.;$$

or,

$$\begin{array}{rr}
a_{(x)}\overline{\rceil}_{t-1}=a_{31}\overline{\rceil}_{4}= & 15\cdot 881\\
a_{(x)}\overline{\rceil}_{2t-1}=a_{31}\overline{\rceil}_{9}= & 12\cdot 063\\
a_{(x)}\overline{\rceil}_{3t-1}=a_{31}\overline{\rceil}_{14}= & 8\cdot 937\\
a_{(x)}\overline{\rceil}_{4t-1}=a_{31}\overline{\rceil}_{19}= & 6\cdot 403\\
\hline
 & 43\cdot 284\\
 & \cdot 002\\
\hline
 & \cdot 086568\\
\mathrm{A}_x=\mathrm{A}_{31}= & \cdot 40235\\
\hline
 & \cdot 31579
\end{array}$$

$$1+a_x=1+a_{31}=\frac{\cdot 31579}{20\cdot 519}=\cdot 01539\times 100=£1.\ 10s.\ 10d.$$

On the above example I note that Mr. Chisholm, in his valuable work, has the following remarks:—

"In the fundamental equation given above, the right-hand side can only express the value of A_x when p multiplies the whole of that side. For the premium for the first t years being p, its value for the whole of life is $p(1+a_x)$; and the increase or decrease at the end of every t years being $p\times q$, its value is evidently

$$p\times q(a_{(x)\rceil t-1}+a_{(x)\rceil 2t-1}+a_{(x)\rceil 3t-1}+\ .\ .\ .\ .\ +a_{(x)\rceil vt-1}).$$

"The writer of the article has not adverted to this, and in the example which follows the rule there is a misconception as to the value of q, where it is made equal to $\cdot 002$ in place of $=\cdot 2$, the increase per pound as assumed.

"The equation and solution should have stood thus:—

$$A_x=p\{(1+a_x)\pm q(a_{(x)\rceil t-1}+a_{(x)\rceil 2t-1}+a_{(x)\rceil 3t-1}+\ .\ .\ .\ .\ +a_{(x)\rceil vt-1})\};$$

by transposition and division,

$$p=\frac{A_x}{1+a_x\pm q(a_{(x)\rceil t-1}+{}_{(x)\rceil 2t-1}+a_{(x)\rceil 3t-1}+\ .\ .\ .\ .\ +a_{(x)\rceil vt-1})};$$

by substitution in the first equation,

$$\frac{M_x}{D_x}=p\,\frac{N_{x-1}\pm q(N_{x+t-1}+{}_{x+2t-1}+N_{x+3t-1}+\ .\ .\ .\ .\ +N_{x+vt-1})}{D_x};$$

from which we obtain

$$p=\frac{M_x}{N_{x-1}\pm q(N_{x+t-1}+N_{x+2t-1}+N_{x+3t-1}+\ .\ .\ .\ .\ +N_{x+vt-1})}."$$

On a careful review of the foregoing, I think it will be found that the misconception as to the value of q is not that of "the writer of the article" in *Jones on Annuities*, but may rather be imputed to Mr. Chisholm, who appears to have confounded the terms of the proposition—viz., the increase of the premium per £1 assured with what he terms "the increase per pound as assumed" (*scil.* in the premium). It is evident that Mr. Chisholm had in his mind the problem which follows (No. 5), the expression for which he has investigated in his work.

Example.—What annual premium should be required to insure £100 on a life aged 31, to be decreased 4*s.* every 5 years, and to remain constant at the end of 20 years?

$$\begin{array}{rr} \text{As in last example} & 2962\cdot 849 \\ M_x=M_{31}\ = & 13770\cdot 543 \\ \hline & 16733\cdot 392 \end{array}$$

$$\text{and } \frac{16733\cdot 392}{N_{x-1}=N_{30}=702266\cdot 223}=\cdot 023827\times 100=£2.\ 7s.\ 8d.\text{ nearly.}$$

$$\begin{array}{rr} \text{Or, as above,} & \cdot 08657 \\ A_x = A_{31} = & \cdot 40235 \\ \hline & \cdot 48892 \end{array}$$

$$\text{and } \frac{\cdot 48892}{1+a_x=1+a_{30}=\ 20\cdot 519} = \cdot 023827 \text{ (as before).}$$

II. A second form of this mode of assurance, which sometimes occurs, is as follows:—

An annual premium of p is to be paid for first t years, p being an arbitrary sum, but one which, in the case of an increasing premium, should be greater than a temporary assurance for a like term; $p \pm q$ for the second term of t years, q also being an arbitrarily fixed sum; $p \pm 2q$ for the third term of t years . . . and $p \pm (v-2)q$ for the $(v-1)$th term, and $p \pm q'$ for the vth term, at which it is to continue constant during the remainder of life: what is the value of q'?

Proceeding as before:—

Benefit term.

An assurance of £1 at death, the present value of which . . . . $\Big\} = A_x$.

Payment terms.

An annuity of £p, previously determined, for the whole of life, the present value of which . . $\Big\} = p(1+a_x)$.

$\pm$ A deferred annuity of q (also first determined), first payment or deduction at end of t years . $\Big\} = q(a_{(x)\rceil t-1})$.

$\pm$ A further deferred annuity of q (as before), first payment or deduction to be made at the end of $2t$ years $\Big\} = q(a_{(x)\rceil 2t-1})$.

$\pm$ &c. &c.

$\pm$ A deferred annuity of q', first payment or deduction at the end of vt years . . . . $\Big\} = q'(a_{(x)\rceil vt-1})$.

Equating, we obtain

$$A_x = p(1+a_x) \pm q(a_{(x)\rceil t-1} + a_{(x)\rceil 2t-1} + \ldots + a_{(x)\rceil (v-1)t-1}) \pm q'(a_{(x)\rceil vt-1});$$

by transposition and division,

$$\pm q' = \frac{A_x - p(1+a_x) \mp q(a_{(x)\rceil t-1} + a_{(x)\rceil 2t-1} + \ldots + a_{(x)\rceil (v-1)t-1})}{a_{(x)\rceil vt-1}};$$

by substitution in the first equation,

$$\frac{M_x}{D_x} = p \cdot \frac{N_{x-1}}{D_x} \pm q \frac{(N_{x+t-1} + N_{x+2t-1} + \ldots + N_{x+(v-1)t-1}) \pm q' \cdot N_{x+vt-1}}{D_x};$$

from which we obtain

$$\pm q' = \frac{M_x - p \cdot N_{x-1} \mp q \cdot (N_{x+t-1} + N_{x+2t-1} + \ldots + N_{x+(v-1)t-1})}{N_{x+vt-1}}.$$

Example.—An annual premium of £1·539 is to be charged for the assurance of £100 on a life aged 31, for the first 5 years, to be increased

to £1·739 for the second 5 years, £1·939 for the third 5 years, and £2·139 for the fourth 5 years: what further increase should be made for the fifth stage, which is to continue constant for the remainder of life?

$$t=5, \qquad v=4, \qquad q=\cdot 002 \text{ per £1 assured.}$$

$$\begin{array}{r}
N_{x-1}=N_{30}=702266\cdot 223 \\
93510 \\
\hline
70226622 \\
35113311 \\
2106799 \\
632040 \\
\hline
10807\cdot 8772
\end{array}
\qquad
\begin{array}{r}
N_{x+t-1}=543540\cdot 812 \\
N_{x+2t-1}=412864\cdot 847 \\
N_{x+3t-1}=305877\cdot 513 \\
\hline
1262283\cdot 172 \\
\cdot 002 \\
\hline
2524\cdot 566344 \\
10807\cdot 8772 \\
\hline
13332\cdot 4435 \\
M_x=M_{31}=13770\cdot 543 \\
\hline
438\cdot 100
\end{array}$$

$$N_{vt-1}=N_{50}=\frac{438\cdot 1}{219141\cdot 5}=\cdot 002\times 100=\cdot 2=4s.;$$

or,

$$\begin{array}{r}
1+a_x=1+a_{31}=20\cdot 519 \\
p \text{ inverted}=\ 93510 \\
\hline
20519 \\
10260 \\
615 \\
180 \\
\hline
\cdot 31574
\end{array}
\qquad
\begin{array}{r}
a_{(x)\overline{t-1}|}=a_{31\overline{4}|}=15\cdot 881 \\
a_{(x)\overline{2t-1}|}=a_{31\overline{9}|}=12\cdot 063 \\
a_{(x)\overline{3t-1}|}=a_{31\overline{14}|}=\ 8\cdot 937 \\
\hline
36\cdot 881 \\
\cdot 002 \\
\hline
\cdot 073762 \\
\cdot 31574 \\
\hline
\cdot 38950 \\
A_{31}=40235 \\
\hline
\cdot 01285
\end{array}$$

$$a_{31\overline{19}|}=\frac{\cdot 01285}{6\cdot 403}=\cdot 002\times 100=\cdot 2, \text{ as before.}$$

The first premium, and the addition at the second, third, and fourth stages respectively, might have been differently chosen, but the values taken were selected because they afford a proof of the first case. This remark applies also to cases 3 and 4.

Example.—An annual premium of £2·38278 is to be charged for an assurance of £100 on a life aged 31, for the first 5 years, to be decreased to £2·18278 for the second 5 years, to £1·98278 for the third five years, and to £1·78278 for the fourth 5 years, what is the decrease which should be made for the fifth stage, and to remain constant for the rest of life?

$$N_{x-1}=N_{30}=702266\cdot223\times\cdot0238278=16733\cdot459$$

As in preceding example, $2524\cdot566$

$$14208\cdot893$$

$$M_x=M_{31} = 13770\cdot543$$

$$-438\cdot350$$

and $\dfrac{-438\cdot350}{N_{vt-1}=N_{50}=219141\cdot5}=-\cdot002\times100=-\cdot2=4s.$

or, $1+a_x=1+a_{31}=20\cdot519\times\cdot0238278=\cdot48893$

As in preceding example, $\cdot07376$

$$\cdot41517$$

$$A_x=A_{31} = \cdot40235$$

$$-\cdot01282$$

and $\dfrac{-\cdot01282}{a_{31\rceil19}=a\,50=6\cdot403}=-\cdot002$ (as before).

III. The third case proposed is one where, the first premium being fixed, it is required to find the rate of increase or decrease for each succeeding stage.

Suppose the premium for the first t years to be p, what premium must be paid for the second, third, &c., series of t years, the premium for the vth stage being constant throughout the rest of life?

This may be stated as follows:—

Benefit term.

An assurance of £1 payable at death, the present value of which . . . . . . $\}=A_x$.

Payment terms.

An annuity of £p, already determined, the present value of which . . . . . . $\}=p(1+a_x)$.

$\pm$ A deferred annuity of q, first payment or deduction at end of t years . . . . . . $\}=q(a_{(x)\rceil t-1})$.

$\pm$ do. do. $2t$ years $=q(a_{(x)\rceil 2t-1})$.

$\pm$ do. do. $3t$ years $=q(a_{(x)\rceil 3t-1})$.

. . . . .

$\pm$ do. do. vt years $=q(a_{(x)\rceil vt-1})$.

Equating,

$$A_x=p(1+a_x)\pm q(a_{(x)\rceil t-1}+a_{(x)\rceil 2t-1}+a_{(x)\rceil 3t-1}+\ldots\ldots+a_{(x)\rceil vt-1});$$

by transposition and division,

$$q=\frac{A_x\mp p(1+a_x)}{a_{(x)\rceil t-1}+a_{(x)\rceil 2t-1}+a_{(x)\rceil 3t-1}+\ldots\ldots+a_{(x)\rceil vt-1}};$$

by substitution in the first equation,

$$\frac{M_x}{D_x}=p\cdot\frac{N_{x-1}}{D_x}\pm\frac{q(N_{x+t-1}+N_{x+2t-1}+N_{x+3t-1}+\ldots\ldots+N_{x+vt-1})}{D_x};$$

from which we obtain

$$\pm q = \frac{M_x \mp p \, . \, N_{x-1}}{N_{x+t-1} + N_{x+2t-1} + N_{x+3t-1} + \ldots . + N_{x+vt-1}}.$$

Example.—An annual premium of £1·539, for an assurance of £100 on a life aged 31, is to be paid for the first 5 years; what must be the increase at the end of 5, 10, 15, and 20 years, the premium for the last stage to be constant during the rest of life?

$$t=5, \qquad v=4.$$

$$M_x = M_{31} = 13770{\cdot}543$$

$$N_{x-1} = N_{30} = 702266{\cdot}223 \times {\cdot}01539 = 10807{\cdot}877 \text{ (as before)}$$

$$2962{\cdot}666$$

$$\begin{aligned} N_{x+t-1} &= N_{35} = 543540{\cdot}812 \\ N_{x+2t-1} &= N_{40} = 412864{\cdot}847 \\ N_{x+3t-1} &= N_{45} = 305877{\cdot}513 \\ N_{x+4t-1} &= N_{50} = 219141{\cdot}537 \end{aligned}$$

$$1481424{\cdot}709$$

$$\text{and } \frac{2962{\cdot}666}{1481424{\cdot}709} = {\cdot}002 \times 100 = {\cdot}2.$$

Otherwise,

$$A_x = A_{31} = {\cdot}40235$$

$$1 + a_x = 1 + a_{31} = 20{\cdot}519 \times {\cdot}01539 = {\cdot}31574 \text{ (as before)}$$

$${\cdot}08661$$

$$\begin{aligned} a_{(x)\overline{t-1}|} &= a_{31\overline{4}|} = 15{\cdot}881 \\ a_{(x)\overline{2t-1}|} &= a_{31\overline{9}|} = 12{\cdot}063 \\ a_{(x)\overline{3t-1}|} &= a_{31\overline{14}|} = 8{\cdot}937 \\ a_{(x)\overline{4t-1}|} &= a_{31\overline{19}|} = 6{\cdot}403 \end{aligned}$$

$$43{\cdot}284$$

$$\text{and } \frac{{\cdot}08661}{43{\cdot}284} = {\cdot}002 \times 100 = {\cdot}2.$$

Example.—An annual premium of £2·38278 for an assurance of £100 on a life aged 31 is to be paid for the first 5 years, what must be the decrease at the end of 5, 10, 15, and 20 years, the premium for the last stage to be constant during the remainder of life?

As before, $N_{x-1} = N_{30} = 702266{\cdot}223 \times {\cdot}0238278 = 16733{\cdot}459$

$$M_x = M_{31} = 13770{\cdot}543$$

$$-2962{\cdot}916$$

and as before, $\dfrac{-2962{\cdot}916}{1481424{\cdot}709} = -{\cdot}002 \times 100 = -{\cdot}2;$

or, as before, $1 + a_x = 1 + a_{31} = 20{\cdot}519 \times {\cdot}0238278 = {\cdot}48893$

$$A_x = A_{31} = {\cdot}40235$$

$$-{\cdot}08658$$

and as before, $\dfrac{-\cdot 08658}{43\cdot 284} = -\cdot 002$ (as before).

IV. The fourth form of this mode of assurance above referred to is one where the premium for the first stage is fixed at p, for the second stage at p_1, for the third stage at p_2 . . . . and for the $(v-1)$th stage at $p_{(v-1)}$; required to find the value of p_v for the vth stage, at which it will remain constant.

The benefit and payment terms in this case are as follows:—

Benefit term.

An assurance of £1 payable at death, the present value of which . . . $= A_x$.

Payment terms.

An annuity of £p for the whole of life, the present value of which . . . $= p(1+a_x)$.

$\pm$ A deferred annuity of $(p_1 \sim p)$, first payment at end of t years . . . $= (p_1 \sim p)\,(a_{(x)\rceil t-1})$.

$\pm$ A deferred annuity of $(p_2 \sim p_1)$, first payment at end of $2t$ years . . . $= (p_2 \sim p_1)\,(a_{(x)\rceil 2t-1})$.

$\pm$ A deferred annuity of $(p_{(v-1)} \sim p_{(v-2)}$, first payment at end of $v-1$ years . $= (p_{(v-1)} \sim p_{(v-2)})\,(a_{(x)\rceil (v-1)t-1})$.

$\pm$ And a deferred annuity of $p_v \sim p_{(v-1)}$, first payment at end of v years . . $= (p_v \sim p_{(v-1)})\,(a_{(x)\rceil vt-1})$.

Equating,

$$A_x = p(1+a_x) \pm (p_1 \sim p)\,(a_{(x)\rceil t-1}) \pm (p_2 \sim p_1)\,(a_{(x)\rceil 2t-1}) \pm \ldots \pm$$
$$\pm (p_{(v-1)} \sim p_{(v-2)})\,(a_{(x)\rceil (v-1)t-1}) \pm (p_v \sim p_{(v-1)})\,(a_{(x)\rceil vt-1});$$

by transposition and division,

$$\pm(p_v \sim p_{(v-1)}) =$$
$$= \frac{A_x - p(1+a_x) \pm (p_1 \sim p)\,(a_{(x)\rceil t-1}) \pm (p_2 \sim p_1)(a_{(x)\rceil 2t-1}) \pm \ldots \pm (p_{(v-1)} \sim p_{(v-2)})\,(a_{(x)\rceil (v-1)t-1})}{a_{(x)\rceil vt-1}}$$

by substitution in the first equation,

$$\frac{M_x}{D_x} = p\frac{N_{x-1}}{D_x} \pm (p_1 \sim p)\frac{N_{x+t-1}}{D_x} \pm (p_2 \sim p_1)\frac{N_{x+2t-1}}{D_x} \pm \ldots . \pm$$
$$\pm (p_{(v-1)} \sim p_{(v-2)})\frac{N_{x+(v-1)t-1}}{D_x} \pm (p_v \sim p_{(v-1)})\frac{N_{x+vt-1}}{D_x};$$

whence we obtain

$$\pm(p_v \sim p_{(v-1)}) =$$
$$= \frac{M_x \pm p.N_{x-1} \pm (p_1 \sim p)N_{x+t-1} \pm (p_2 \sim p_1)N_{x+2t-1} \pm \cdots \pm (p_{(v-1)} \sim p_{(v-2)})N_{x+(v-1)t-1}}{N_{x+vt-1}}.$$

Example.—For an assurance of £100 on a life aged 31, the premium for the first stage of 5 years is to be £1·539, for the second stage £1·739, for the third stage of 5 years £1·939, for the fourth stage of 5 years £2·139: what further increase should be made for the fifth stage, and to continue for the rest of life?

$$M_x = M_{31} = \quad 13770 \cdot 543$$

$$N_{x-1} = N_{30} = 702266 \cdot 223 \times \cdot 01539 = \quad 10807 \cdot 877 \text{ (as before)}$$

$$(p_1 - p) = \cdot 01739 - \cdot 01539 = \cdot 002$$

$$N_{x+t-1} = N_{35} = 543540 \cdot 812 \times \cdot 002 = \quad 1087 \cdot 082$$

$$(p_2 - p_1) = \cdot 01939 - \cdot 01739 = \cdot 002$$

$$N_{x+2t-1} = N_{40} = 412864 \cdot 847 \times \cdot 002 = \quad 825 \cdot 730$$

$$(p_3 - p_2) = \cdot 02139 - \cdot 01939 = \cdot 002$$

$$N_{x+3t-1} = N_{45} = 305877 \cdot 513 \times \cdot 002 = \quad 611 \cdot 755$$

$$13332 \cdot 444$$

$$438 \cdot 099$$

and

$$N_{x+vt-1} = N_{50} = \frac{438 \cdot 099}{219141 \cdot 709} = \cdot 002 \times 100 = \cdot 2;$$

or,

$$A_x = A_{31} = \quad \cdot 40235$$

$$1 + a_x = 1 + a_{31} = 20 \cdot 519 \times \cdot 01539 = \quad \cdot 31574$$

$$a_{(x)\overline{\rceil}t-1} = a_{31\overline{\rceil}4} = 15 \cdot 881 \times \cdot 002 = \quad \cdot 03176$$

$$a_{(x)\overline{\rceil}2t-1} = a_{31\overline{\rceil}9} = 12 \cdot 063 \times \cdot 002 = \quad \cdot 02413$$

$$a_{(x)\overline{\rceil}3t-1} = a_{31\overline{\rceil}14} = 8 \cdot 937 \times \cdot 002 = \quad \cdot 01787$$

$$\cdot 38950$$

$$\cdot 01285$$

$$a_{(x)\overline{\rceil}vt-1} = a_{31\overline{\rceil}19} = \frac{\cdot 01285}{6 \cdot 403} = \cdot 002 \times 100 = \cdot 2.$$

Example.—For an assurance of £100 on a life aged 31, the premium for the first 5 years is to be £2, for the second 5 years £1, for the third 5 years £1. 10*s.*, for the fourth 5 years £1. 5*s.*; what further increase or decrease should be made for the fifth stage, and to continue during the remainder of life?

$$N_{x-1} = N_{30} = 702266 \cdot 223 \times \cdot 02 \quad \ldots \quad = 14045 \cdot 324$$

$$(p_1 - p) = \cdot 02 - \cdot 01 = 01$$

$$N_{x+t-1} = N_{35} = 534540 \cdot 812 \times \cdot 01 \quad = 5435 \cdot 408$$

$$(p_2 \sim p_1) = \cdot 01 \sim \cdot 015 = \cdot 005$$

$$N_{x+2t-1} = N_{40} = 412864 \cdot 847 \times \cdot 005 \quad \ldots \quad = 2064 \cdot 324$$

$$(p_3 - p_2) = \cdot 015 - \cdot 0125 = \cdot 0025$$

$$N_{x+3t-1} = N_{45} = 305877 \cdot 513 \times \cdot 0025 = 764 \cdot 694$$

$$16109 \cdot 648$$

$$6200 \cdot 102$$

$$9909 \cdot 546$$

$$M_x = M_{31} = \quad 13770 \cdot 543$$

$$3860 \cdot 997$$

and

$$N_{x+4t-1} = N_{50} = \frac{3860 \cdot 997}{219141 \cdot 709} = \cdot 01762 \times 100 = 1 \cdot 762;$$

or,

$$\begin{array}{lllll}
1+a_x & =1+a_{31} & =20\cdot519\times\cdot02 & = \;.\;\;. & =\cdot41038 \\
-a_{(x)\rceil t-1} & =a_{31\rceil 4} & =15\cdot881\times\cdot01 & =15881 & \\
+a_{(x)\rceil 2t-1} & =a_{31\rceil 9} & =12\cdot063\times\cdot005 \;.\;\;.\;\;. & & =\cdot06031 \\
-a_{(x)\rceil 3t-1} & =a_{31\rceil 14} & =\;8\cdot937\times\cdot0025 & =\underline{02234} & \\
 & & & & \overline{\cdot47069} \\
 & & & & \cdot18115 \\
 & & & & \overline{\cdot28954} \\
 & & & A_x=A_{31}\;= & \cdot40235 \\
 & & & & \overline{\cdot11281}
\end{array}$$

and $\dfrac{\cdot11281}{a_{(x)\rceil 4t-1}=a_{30\rceil 14}=6\cdot403}=\cdot01762$ (as before).

V. The fifth and last case proposed to be treated is one where the first premium is increased or decreased by a fixed proportion at each of the succeeding stages; thus:—

An annual premium of p is to be paid for the first t years, to be increased or decreased by $\frac{p}{q}$ for the second t years, by $2.\frac{p}{q}$ for the third t years . . . . and by $(v-1)\frac{p}{q}$ for the vth t years; to find the value of p.

The statement of this case is as follows:—

Benefit term.

An assurance of £1 for life, the present value of which . . . . $\Big\}=A_x$.

Payment terms.

An annuity of £p for the whole of life, the present value of which . . . . . . $\Big\}\; p(1+a_x)$.

$\pm$ A deferred annuity of $\frac{p}{q}$, first payment or deduction at the end of t years . . . . $\Big\}\; \frac{p}{q}\cdot a_{(x)\rceil t-1}$.

$\pm$ A deferred annuity of $\frac{p}{q}=\left(2\frac{p}{q}-\frac{p}{q}\right)$, first payment or deduction at the end of $2t$ years . $\Big\}\; \frac{p}{q}\cdot a_{(x)\rceil 2t-1}$.

. . . . .

$\pm$ A deferred annuity of $\frac{p}{q}=\left((v-2).\frac{p}{q}-(v-1).\frac{p}{q}\right)$, first payment or deduction at the end of vt years $\Big\}\; \frac{p}{q}\cdot a_{(x)\rceil vt-1}$.

Equating,

$$A_x=p\left\{(1+a_x)\pm\frac{1}{q}\left(a_{(x)\rceil t-1}+a_{(x)\rceil 2t-1}+\;.\;.\;.\;.\;+a_{(x)\rceil vt-1}\right)\right\};$$

by substitution and division,

$$p=\frac{A_x}{(1+a_x)\mp\frac{1}{q}\,a_{(x)\rceil t-1}+a_{(x)\rceil 2t-1}+\;.\;.\;.\;.\;+a_{(x)\rceil vt-1}};$$

by substitution in the first equation,

$$\frac{\mathrm{M}_x}{\mathrm{D}_x}=p.\left\{\frac{\mathrm{N}_{x-1}\pm\frac{1}{q}(\mathrm{N}_{x+t-1}+\mathrm{N}_{x+2t-1}+\ldots+\mathrm{N}_{x+vt-1})}{\mathrm{D}_x}\right\};$$

from which we get

$$p=\frac{\mathrm{M}_x}{\mathrm{N}_{x-1}\mp\frac{1}{q}(\mathrm{N}_{x+t-1}+\mathrm{N}_{x+2t-1}+\ldots+\mathrm{N}_{x+vt-1})}.$$

Example.—What annual premium should be required during the first five years to insure £100 on a life aged 31, the annual premium to increase 4*s.* per £1, or one-fifth, every 5 years, and to remain constant at the end of 20 years?

$$t=5,\qquad v=4,\qquad \frac{1}{q}=\cdot 2.$$

$$\begin{array}{rr}
\mathrm{N}_{x+t-1}=\mathrm{N}_{35}= & 543540\cdot812 \\
\mathrm{N}_{x+2t-1}=\mathrm{N}_{40}= & 412864\cdot847 \\
\mathrm{N}_{x+3t-1}=\mathrm{N}_{45}= & 305877\cdot513 \\
\mathrm{N}_{x+4t-1}=\mathrm{N}_{50}= & 219141\cdot537 \\
\hline
 & 1481424\cdot709 \\
\frac{1}{q}= & \cdot 2 \\
\hline
 & 296284\cdot9418 \\
\mathrm{N}_{x-1}=\mathrm{N}_{30}= & 702266\cdot223 \\
\hline
 & 998551\cdot165
\end{array}$$

$$\mathrm{M}_x=\mathrm{M}_{31}=\frac{13770\cdot543}{998551\cdot165}=\cdot01379\times100=£1.\ 7s.\ 7d.,$$

$$\text{and}\ \frac{1}{q}=\cdot2\ \therefore\ \cdot2\times\cdot002758\times100=5s.\ 4d.\ \text{nearly};$$

or,

$$\begin{array}{rr}
a_{(x)\overline{t-1|}}=a_{31\overline{4|}}= & 15\cdot881 \\
a_{(x)\overline{2t-1|}}=a_{31\overline{9|}}= & 12\cdot063 \\
a_{(x)\overline{3t-1|}}=a_{31\overline{14|}}= & 8\cdot937 \\
a_{(x)\overline{3t-1|}}=a_{31\overline{19|}}= & 6\cdot403 \\
\hline
 & 43\cdot284 \\
\frac{1}{q}= & \cdot2 \\
\hline
 & 8\cdot6568 \\
1+a_x=1+a_{31}= & 20\cdot519 \\
\hline
 & 29\cdot176
\end{array}$$

$$\mathrm{A}_x=\mathrm{A}_{31}=\frac{\cdot40235}{29\cdot176}=\cdot01379\times100=£1.\ 7s.\ 7d.$$

Example.—What annual premium should be required during the first 5 years to insure £100 on a life aged 31, the annual premium to be decreased 4*s.* per £1, or one-fifth, every 5 years, and to remain constant at the end of 20 years?

$$N_{x-1}=N_{30}=702266{\cdot}223$$

$$\text{—as in preceding example . . . } \underline{296284{\cdot}942}$$

$$405981{\cdot}281$$

$$\text{and } M_x=M_{31}=\frac{13770{\cdot}543}{405981{\cdot}281}={\cdot}033919\times100=£3{\cdot}3919$$

$$\text{and } \frac{1}{q}={\cdot}2 \therefore {\cdot}2\times{\cdot}006784\times100=13s.\ 9d. \text{ nearly.}$$

Or,

$$1+a_x=1+a_{31}=20{\cdot}519$$

$$\text{—as in preceding example . } \underline{8{\cdot}657}$$

$$11{\cdot}862$$

$$\text{and } A_x=A_{31}=\frac{{\cdot}40235}{11{\cdot}862}={\cdot}033919 \text{ (as before).}$$

In conclusion, it will be obvious that the values or ratio assigned to p and q, as well as the terms t and v, may, within certain limits, be varied at pleasure.

Apologising for the length of this communication,

I am, Sir,
Your obedient servant,
SAMUEL L. LAUNDY.

Eagle Life Office,
24th March, 1862.

INSTITUTE OF ACTUARIES.

PROCEEDINGS OF THE INSTITUTE.

Fifteenth Annual General Meeting, Saturday, 7th June, 1862.

Robert Tucker, Vice-President, in the Chair.

Mr. John Hill Williams, one of the Honorary Secretaries, read the notice convening the meeting, the minutes of the preceding ordinary meeting, and the following Report :—

"The Council have again the pleasure of submitting their Annual Report to the members of the Institute.

"The number of members now on the register is 167—twelve more than that last reported. This number is made up of 48 fellows, 24 official associates, and 95 associates.

"The income of the year is £414. 6*s.* 6*d.*, the expenditure £373. 1*s.* 5*d.*, and the assets £455. 15*s.* 3*d.* For details, and for the relation which these figures bear to those of former years, the Council refer to the Report of the Assistant-Secretary, a document which is open at all times to the inspection of the members, and in which all the information desired can be found.

"The following papers have been read during the Session, viz.:—

"'Observations on the tables of mortality experience of the Scottish Amicable Society.' By William Spens.

"'Observations on Gompertz's law of mortality and the dependence between it and Simpson's rule for finding the value of an annuity on three lives.' By W. S. B. Woolhouse.

"'On the tendency of some systems of distribution of surplus to defeat the object of life assurance.' By James Terry.

"'On the principles on which the funds of Life Assurance Societies should be invested.' By Arthur Hutcheson Bailey.

"'On the statistics of first and subsequent marriages amongst the families of the peerage, considered specially with reference to the calculation of premiums for assurances against issue.' By Archibald Day.

"'On the probable future of the rate of interest.' By Wm. Newmarch, F.R.S.

"To all engaged in the pursuits of an actuary, these papers, no less than those which have preceded them, will be found full of interest and instruction. It will be seen that they fully sustain the useful and practical character which has hitherto distinguished the publications of the Institute, and which have rendered them of so much value, not only to the student, but to the more matured practitioner."

An abstract of the receipts and payments for the financial year ending 31st March, 1862, was then read (*see* p. 300).

The Report having been unanimously adopted, the meeting proceeded to the election of officers for the ensuing year, Mr. Newbatt and Mr. Clirehugh being appointed scrutineers. The following gentlemen were unanimously elected :—

President.

CHARLES JELLICOE.

Vice Presidents.

ARTHUR HUTCHESON BAILEY.
SAMUEL BROWN.
PETER HARDY, F.R.S.
ROBERT TUCKER.

Treasurer.

JOHN LAURENCE.

Honorary Secretaries.

JOHN REDDISH.
JOHN HILL WILLIAMS.

Mr. John Coles and Mr. James Terry were reappointed Auditors; and Mr. William King, of the English and Scottish Law Life, was added to their number.

The Chairman, in the unavoidable absence of the President, returned thanks for the re-election of the President, Vice-Presidents and other officers. He was sure that the paucity of the attendance was not by any means to be taken as a want of zeal or good feeling in the members towards the Institute. He knew the President and other officers who had taken such lively interest in the welfare of this Institute would not relax their efforts, but would go on conducting it to the best of their ability.

Mr. Reddish also acknowledged the compliment on behalf of himself and Mr. Williams.

Mr. Newbatt proposed that the thanks of the meeting be given to the retiring President, Vice-President, Council, Officers, and Auditors, for their services during the past year. He should hardly have been tempted to add one word to this resolution, which was necessarily of a formal character, but he thought it desirable to offer a few remarks, inasmuch as the members were indebted in a great degree to these officers for the importance of the Institute, and for having given to it that high position which it had reached, both as a scientific body and as the exponent of the principles which they all met together from time to time to advocate and advance. He felt sure that their efforts, which had been so successful in the past, would not be wanting in the future to add even more laurels than had been already gained. He was much gratified that Mr. Bailey had been elected a Vice-President, and he was sure that election would be acceptable to all the members.

Mr. Lodge seconded the motion, which was passed with unanimity.

Mr. Samuel Brown returned thanks, and said they all felt deeply interested in the prosperity of the Institute with which they had been connected for so many years, feeling, as they did, that not only was the status of an actuary greatly strengthened by the discussions which took place in that room, but also that his social qualities were very much improved; and the facilities for carrying on their business were also greatly increased. The officers were deeply grateful for the kind support the members had always rendered.

Votes of thanks were also passed to the Scrutineers and the Chairman, and the meeting concluded.

INSTITUTE OF ACTUARIES.

Dr. *Abstract of Receipts and Payments for the Year ended 31st March, 1862.* Cr.

RECEIPTS.

				£ s. d.
April 1, 1861. To balance brought forward				236 0 8
Subscriptions for 1860–61 (arrears)				4 4 0
Subscriptions due for 1861–62, viz.:—				
48 Fellows ... 36 Town	at £3 3 0		£113 8 0	
12 Country	„ 2 2 0		25 4 0	
24 Official Associates .. 21 Town	„ 3 3 0		66 3 0	
3 Country	„ 2 2 0		6 6 0	
95 Associates ... 75 Town	„ 2 2 0		157 10 0	
1 „	„ 1 1 0		1 1 0	
19 Country	„ 1 1 0		19 19 0	
			389 11 0	
Subscriptions of Members not paid, viz.:—				
1 Fellow, Town	at £3 3 0	£3 3 0		
1 „ Country	„ 2 2 0	2 2 0		
2 Associates, Town	„ 2 2 0	4 4 0		
1 Associate, Country	„ 1 1 0	1 1 0		
			10 10 0	
				379 1 0
Examination Fees (2 at £5. 5*s.*)				10 10 0
Dividends on Messenger Legacy (2 years)				11 9 2
Balance of Petty Cash (paid into Bank)				4 19 0
Sundries				4 3 4
				£650 7 2
March 31, 1862. To Balance brought down				£250 17 7

Note.—The Assets of the Institute, on the 31st March, 1862,

consisted of £211. 1*s.* 10*d.* 3 per cent. Consols, which cost	204 17 8
Cash	250 17 7
Books in Library, say	304 4 9
Total	£760 0 0

12, St. James's Square, London,
April, 1862.

PAYMENTS.

1861–62.	£ s. d.	£ s. d.
By Rent	75 0 0	
Salaries	103 15 0	
Journal	104 15 6	
Stationery and Printing	21 17 0	
Ordinary Meetings	16 2 2	
Lighting	4 4 8	
Coals (2 years)	15 6 0	
Postage and Receipt Stamps	6 8 0	
Library	2 7 0	
Advertising Examinations	4 13 6	
Diagram Paper (100 yards)	3 10 0	
Miscellaneous	15 2 7	
Total of general expenses		373 1 5
Purchase of £12. 5*s.* 8*d.* Stock, 3 per cent. Consols, cost		11 9 2
Amount of Messenger Prize		10 0 0
Balance of Petty Cash (paid into Bank)		4 19 0
		£399 9 7
March 31, 1862. Balance carried forward		250 17 7
		£650 7 2

Examined and approved:—

John Coles,
Edward Cutbush,
James Terry, } *Auditors.*

THE

ASSURANCE MAGAZINE,

AND

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the Forms under which Barrett's Method is presented, and on Changes of Words and Symbols. *By* Professor De Morgan.

THE mode of using Barrett's method has now something of farrago about it: a little discussion may be useful, were it only to awaken attention to the confusion arising from changes made without adherence to the recognised laws of perspicuity.

The farrago to which I have alluded arises out of a struggle between the *annuity,* as commonly understood, and what I shall continue to call the *annuity due.* According to the technical meaning of the word *annuity,* the payment begins to *grow* at the moment of the grant: thus, a yearly annuity for 12 years, deferred 20 years, begins action at the end of 20 years, and grows the first payment in the 21st year. I shall vary the phrase used in my "Essay on Probabilities" (*Cab. Cycl.*) by allowing the words "*annuity due* for x years" to signify an annuity of x *payments,* the first immediately. Thus an annuity for 12 years, now granted, will in a year become an annuity *due,* also for 12 years.

The value of a premium, at the moment of contract, is always that of an *annuity due,* for which *premium* might be made the proper technical term, if people chose: thus an annuity due of seven years might be called a premium of seven years. But the

word *premium* is so firmly fixed to the notion of *something bought with it,* that the proposal might be distasteful as a substitute for the term *annuity,* which is not necessarily the price of anything.

Many writers speak of an annuity as constituted by a number of yearly payments; and would therefore say that a premium is a kind of annuity. Ambiguities arise from using a word which implies a portion of time, without reference to the usual, and therefore proper, dependence of the meaning upon the commencement and termination of the period. Thus a *year* and a *century* are often spoken of as portions of time merely, as in "the year ending June 5," and "the century 1762–1861," which form parts of two years and of two centuries, in the more common meanings of the words. There is not much to find fault with here; but when we deal with the word *annuity* in the same way, we may easily be led into such errors as writing 9 for 8, or 8 for 9, which it is one great business of an actuary's life to avoid. Making the word annuity mean the two different things is dangerous, because the two things are so alike: had they been very different, no mischief would have arisen. If chain armour, mail made of interlaced rings—*annus* meaning really a *ring*—had been called *annuity,* the actuary would not have been injured by *this* double meaning of the word. Slight changes of meaning are dangerous for the very reasons which prevent any comparison from being satirical except when gross exaggeration is used. Some time ago the Admiralty encouraged the introduction into the navy of the word *declination,* to signify the *dip* of the magnet—the seamen having declination already in common astronomical use. My friend the late Lieut. Raper, a man far too high in scientific navigation to approve of such abuse of language, wrote a satire on the change, in which he recommended the Lords to try how inverting the names of the fore-topsail and the jolly boat would work. He thought it might be useful, if the seamen were to be trained in ambiguous terms, to accustom them to bold innovations—so that they might learn not to be surprised at hearing that a ship coming into harbour under close-reefed jolly boat sent the fore-topsail on shore with the purser's bags for fresh beef. But the satire was lessened by the very obvious consideration that the men might get used to this change in a couple of days. But let the foremast and mainmast change names, and it would be a very different thing. It is the greatest aggravation, and therefore the farthest from excuse—of trifling with names—or with symbols, which *are* names—that the ambiguity of language is applied to things which are very much

alike. Anything* but this: call Epsom salts and laudanum by one name who likes; but not Epsom salts and oxalic acid. There would be little danger of anyone taking the *wrong* Epsom salts under the first nomenclature; but every wise man would shun both under the second.

The first innovator in this matter was, I believe, Mr. Sang, whose admirable volumes (1841, 1859), or rather whose first volume, has every annuity in it of the *due* or *premium* kind. The second volume has, for whole-life annuities, both kinds, distinctly separated as "Annuity first payment now," "Annuity first payment one year hence." I am not criticising the plan of the first volume, but only the misuse of the word *annuity*. The short or temporary annuity is wanted in most cases—by me, ninety-nine times out of a hundred—for calculation of a premium. Mr. Sang's plan, for a work having the bulk of its annuities temporary, is convenient; but the use of the word *annuity* is a blemish, and gives risk of error. Those who are going to make frequent use of the book are in no danger; they will easily bear in mind that the annuity is in all cases *due*—that every 10 years' annuity, for instance, is £1 now, and £1 at the end of each of nine years to come. But a casual user of the volume does not expect, and cannot reasonably be expected to expect, that he should be obliged to examine narrowly into all the technical terms he finds, to see whether his author adheres to their usual meanings. Such a person might chance to hear that the work in question contains annuities payable up to 50, 55, &c., and he finds a page of "*Annuities* of which the *payment* is to cease on the attainment of the undermentioned ages." He wants an annuity for age 20, ceasing at 70, and he finds 21·965. The last payment is made at the age 69, and the first now. This is not what was wanted, and is not what the words usually mean. According to common language, technical or not, when a payment is to "cease at the age 70," the last payment is made at that age.

The two systems might be distinguished as the *initial* and the *terminal* systems. The same moment of time is both the end of one year and the beginning of another. The ordinary annuity has its moment of payment reckoned as the end of a year; the annuity *due*, or premium, has its moment of payment reckoned as the

* The record of stories which have taken rise in resemblance of surnames would be a curious book. When looking after the history of Barrett, I had to inquire into a rumour of his having been assisted in his calculations by Griffith Davies. In this there was no truth whatever; but Mr. Davies remembered teaching Mr. *Berry*, who was in the Hope Office with Barrett, and whom he did assist in official calculations after Barrett had left.

beginning of a year. When logarithms are given, as in Mr. Sang's work, the scale is, I think, decidedly turned in favour of the initial, or annuity-due, system. But, were it otherwise, putting the casual user out of the question and speaking only of one who wants it more frequently, he must be a Sybarite, a man whose rest is broken by a crumbled rose-leaf, who would take serious exception to a variation so easily remembered and allowed for, and so permanently adhered to. These volumes are a Carlisle three-per-cent. paradise.

I now come to Barrett's tables; the D and N tables, commutation tables, or whatever they are at last to be called. I should like to retain Barrett's name, but the alterations in the form of the *tables* render this impracticable. The *method*, however, ought to remain attached to Barrett's name. The original form of the tables, as given by him, is *initial*. The columns are A, B, C, in black letter. The number living at the age x being l_x, r the amount of £1 in one year, and n the oldest age of the tables, Barrett's forms are

$$A_x = l_x r^{n-x}, \quad B_x = A_x + A_{x+1} + \ldots \qquad C_x = B_x + B_{x+1} + \ldots$$

Hence $B_x : A_x$ is the annuity due, or the premium, and $B_{x+1} : A_x$ is the common annuity.

Griffith Davies, in his tract of 1825, gave his own alteration into terminal forms. This work, we are told, was put together in a hurry, out of larger materials, on the preliminary announcement of Mr. Babbage's* work, which appeared in 1826. The columns are D, N, S, M, R. The column C, used in calculating M, but of little independent use, was introduced into the statement of results by myself.

This D and N system is terminal; and further, D_x is $l_x r^{-x}$ or $l_x v^x$. We have $N_x = D_{x+1} + \ldots$, and $S_x = N_x + \ldots$; so that the common annuities are N_x and S_x divided by D_x, the annuities *due* requiring N_{x-1} and S_{x-1}.

Which of these two systems is the best? By mathematical analogy no doubt the initial system would be preferred. The same reasons, well known to students of the calculus of differences, which make it imperative to denote $D_0 + \ldots + D_x$ by ΣD_{x+1}, not ΣD_x, would also require that $D_x + D_{x+1} + \ldots$, and not $D_{x+1} + \ldots$,

* William Morgan, in his *Rise and Progress of the Equitable Society*, 1828, made some remarks on Mr. Babbage's account of the Equitable Society. Francis Baily criticised these remarks in the *Times*, June 26, 1828; and Morgan replied July 1, 1828. These letters ought to be preserved, and I have accordingly transmitted my copies to the Editor of this Journal, for publication.

should be denoted by SD_x, or whatever symbol of summation should be used. Again, for the computer's convenience, I should also be inclined to prefer this initial system, on the principle that in life assurance, which much exceeds annuity business in amount, the annuity due, or premium, is of more importance than the simple annuity. But I am now speaking of the creation of a system. Whether it would be worth while to disturb a system in use* for the difference between the two, introduces other considerations. I discussed this question with myself when I wrote my papers on the subject: and my conclusion was that it was *not* worth while; or, rather, that the inconvenience of conflicting systems would outweigh the slight theoretical and practical advantage both.

Dr. Farr was the introducer of the initial system into the commutation tables. His announcement of the change is made in the following words (*Sixth Report of the Registrar-General,* 1845, p. 593) :—

> "This derangement [Davies's alteration of Barrett's arrangement] of the simple relations of the respective columns for the supposed and not real convenience of the computer, always appeared to me the reverse of an improvement; and upon finding that the columns y and $S.y$ in Duvillard's table, and the original columns A and B in Barrett's table, stand in the same relation to each other as they now stand in the English tables, I adopted that arrangement without hesitation."

The justification of the change is twofold. First, Dr. Farr thought that the established system might be changed for the better; secondly, he found authority in Duvillard's table, which is hardly known, and in Barrett's table, which was known only by specimen, Davies's form being the only one ever *used.* The question of convenience or inconvenience, arising out of the simultaneous currency of conflicting systems, appears to have been thought not worth consideration. But the change, *when properly made,* is so easily used, that these remarks may pass only as warning to future writers not to vary existing methods without at least the attempt to take into account the question of change *as change,* and to justify themselves on this point. Such justification may be possible; but it is imperatively required.

The change was *not* properly made. Griffith Davies, with the tact of a mathematician, abandoned Barrett's *letters* when he abandoned Barrett's *use* of them. Dr. Farr retained N and S, altering the sense in which they are used. This is equivalent to

* We are reminded of the Greek philosopher, who maintained that life and death are all one. "Why then do you not kill yourself?" asked an opponent. "Because it *is* all one," replied the sage."

the alteration—and, worst of all, the slight alteration—of the meaning of a *name.* On this point I hold it unnecessary to say another word.

Were I now to write a work on the subject, I should use the letter I to distinguish the initial system; the simple letter, as of old, denoting the terminal system. I should have a new type cut, an N with the first stroke made into an I, to denote the N of the initial system. The S is so litttle used that IS might be sufficient. The C, M, R, have, fortunately, not been meddled with.

Mr. Chisholm, in his most useful work, *Commutation Tables,* &c., 1858, has adopted the initial system. His warnings are clear enough to those who examine before using, with one exception, which shows how apt slight differences are to be held no differences at all. He says (p. xlii.) that "the D and N columns for joint lives thus formed, being identical in construction with the preparatory tables of Mr. Jones on the Carlisle basis, the two sets of tables were carefully read over together . . . ." The words "identical in construction" are not correctly applied to two tables which always differ in tabular result throughout one of the two columns which compose them. This sentence, which comes very conspicuously at the end of a half page of print, once nearly led me into serious error. But Mr. Chisholm's work, like Mr. Sang's, is above slight criticism, both by the quality and the quantity of its matter.

Mr. Chisholm's work offers an incidental example of the superior legibility of figures with heads and tails. The type most used is large and even; but, from the necessity of the case, the commutation tables for single lives are in a much smaller type, which happens to have heads and tails. I think anyone who looks at the two will see that the smaller figures project themselves into the eye, while the larger ones must be picked up by the eye.

To return to the subject. It is much to be hoped that in all new undertakings, when any apparently desirable change presents itself, severe examination will be given to the question whether the alteration, be it what it may in itself, is an advantageous addition to what actually exists. But even this is of small importance compared with the warning of history, which tells us not to meddle with names and change their meanings. *Extension* is allowable, when advantageous; that is, inclusion of the old meaning, *and more.* But mere change, whatever may be right, is always wrong.

It may be worth remembrance that since D_0 is always 1000 . . ., the *initial* form immediately shows itself by the first two Ns, N_0 and N_1, differing in only one of their digits.

While on the subject of changes, I may notice a tendency which prevails to substitute one word for two, under the idea of abbreviation. If letters be saved, there is advantage gained in writing. If syllables be saved, there is often—not always—advantage gained in speaking. But if neither letters nor syllables be saved, there can be no advantage, unless some awkward sound be got rid of. What then shall we say to the word *predecease,* in the sense of *die before,* with one more letter, as many syllables, and an ugly double sibilant? The word *decease* itself is a disgrace to our language: it must have been made* by some one who imagined his word was connected with *cessare* instead of *cedere.* I wish some of our purists would write *decess,* as in *process, recess, access, success, excess,* from the same verb. In the seventeenth century, *decession* is noted as a word coming in. The etymological objection, however, might be overlooked if advantage were gained. In any case the word should be *predecede,* and *precede* would do; for an actuary's order of going almost always relates to death. But surely *die before* is better than either; and it allows its correlative *die after.* Are we also to have *post-deceasing?* To my ear *predecease,* as an active verb, conveys the idea of previous slaughter; and a man who precedeased his father is one, I suppose, who killed his father before he killed his mother.

When two words become a phrase, they are really one word, both in meaning and pronunciation. Etymologists are now agreed, I believe, that every syllable is a word, originally. The Chinese language, commonly called monosyllabic, is only so in this respect, that all its syllables have kept their separate meanings: the Chinese speaker runs his significant syllables together, and makes polysyllabic words, just as we do. If the phrase *annuity due* were used, it would be one word immediately.

Again, the introduction of a monosyllable, as more easy to

* The first notorious place in which I find it is in the authorised version of the Bible; but it also appears in the older Protestant versions. Omitting Isaiah xxvi. 14, which those who compare the Septuagint with the English will suspect of having a difficulty in the Hebrew, it is seen in Matth. xxii. 25, Luke ix. 31, 2 Pet. i. 15. In the first passage ὁ πρωτος γαμησας ἐτελευτησε is translated "the first, when he had married a wife, deceased." And here the Greek is really *came to an end, ceased off;* and here Tyndale (1534) and his followers have the neuter verb *deceased.* But in the other two passages, in which the substantive *decease* of our version is the translation of ἐξοδος, *departure,* the previous Protestant versions have *departing,* while the Rhemish (Roman, 1582) version has *decease.* It may be suspected that Tyndale meant *decessatio,* and the Rhemish translator *decessus,* and that our translators adopted both.

Shakspere has one use of the word: he makes Gower reproach Ancient Pistol with mocking at a memorial of "predeceased valour": and down goes the word into dictionaries as common English of Shakspere's time. Gower is making rather a stilted speech, and it may be surmised that the dramatist intended to represent him as inventing the word for his own use. If so, this is far from being the only case in which the English peculiar to a comic character has been made the English of the poet's contemporaries.

pronounce than a dissyllable, may be a mistake. When a monosyllable is long and emphatic, it very often *is* a dissyllable. When we say *mill,* unless we use great effort, the removal of the tongue from the palate before the sound has ceased makes *mill-e,* and gives the second syllable which the French recognise in *cel-le,* which English learners generally take for a monosyllable. Consequently, *miller,* pronounced without the *r,* as usual, is an easier word than *mill:* the second syllable is easier than the pull-up by which it is avoided. If the call-boys in the steam-packets were made to say "Ease," "Stop," "Back," instead of "Ease her," "Stop her," "Back her," they would soon learn the English truth of a Greek pun. They would find out that their new monosyllables are really dyssyllables, and would be glad to restore the old vowel. There is a Joe Miller about a man who heard a countrywoman call her son Hookey, and asked what his name was. "Sir," was the answer, "his name is *Hugh,* but we call him *Hookey* for short." The story-maker set this down as laughable for ignorance, but the good woman was right. And this is the true explanation of Willy and Charley, and a host of other abbreviations.

These things are not out of place when we are on the subject of word-making and word-changing; and for this reason, that some word-makers and word-changers seem not to know them.

There is one point about the printing of commutation tables which deserves notice and discussion. The N and S have so many figures before the decimal point that when, as often happens, only a few are wanted, the selection is perplexing—especially when the difference between two Ns in widely-separated parts of the table is wanted. We need a second decimal point, to separate higher from lower integers. Thus a separator for the thousands would be convenient: if this were used, Chisholm's N_0, at 4 per cent., would be printed 152,816·4338. In doing this we return to an old practice. Before the point or comma became the separator of integer and fraction, it was extensively used as an integer separator. The earliest tables of logarithms gave integer logarithms. In Brigg's *Arithmetica Logarithmica,* for instance, we have

$$\log 10 = 1,\ 00000,\ 00000,\ 0000$$
$$\log 12 = 1,\ 07918,\ 12460,\ 4762$$

It is not true that the first of the commas is our decimal point: Brigg's logarithm of 10 is a hundred million millions. When he wants a separator of fractions, he uses another method. Thus, in the *Lectori Salutem,* he wants the square and fourth roots of $85\frac{1}{3}$, and he gives them thus:—"Latus $85\frac{1}{3}$ est $9\underline{\,^{237604307}}$, et hujus

lateris latus est $3\frac{03934274}{}$ cui numero æquatur $ll85\frac{1}{3}$." It would, perhaps, be desirable that the integer separator should be a semi-colon formed with two commas; so that a casual fracture would certainly leave a comma, which would not be confounded with the decimal point.

The Letters from Francis Baily and William Morgan, referred to herein by Professor De Morgan, are as follows:—

EQUITABLE SOCIETY.

To the Editor of the Times.

SIR,—A pamphlet has lately been published by Mr. Morgan, containing (amongst other things) some remarks on what he is pleased to call "the late misrepresentations respecting the rules and practice of the Equitable Society;" and in the chapter which he has appropriated to this subject he has passed some very unmerited censures, and made many unfounded remarks, on the recent production of my friend Mr. Babbage, the present Lucasian Professor of Mathematics at Cambridge. Mr. Babbage is now making a tour on the continent, and the time of his return to this country is somewhat uncertain. But, as he honoured me with the perusal of his work prior to its being sent to the press, and was pleased to consult me in its progress, I consider myself in some measure called upon, under existing circumstances, to step forward on the present occasion to advocate his cause during his absence, not only by denying that spirit of misrepresentation of which he has been thus falsely accused, but also by challenging Mr. Morgan to a proof of the principal charges which he has thought proper to make, leaving to Mr. Babbage the task of answering more at length, on his return, the whole subject of Mr. Morgan's pamphlet, should he think such a step requisite or necessary.

Mr. Morgan commences his attack on Mr. Babbage, in page 52, by accusing him of stating that the Equitable Society has a large unfunded capital, which it conceals from the knowledge of the public. This charge is so utterly unfounded, and so totally at variance with any opinion that Mr. Babbage has expressed on the subject, that I am quite at a loss to conceive where Mr. Morgan could have met with any passage tending towards such a meaning, or that could even be forced into so opprobrious a construction.

Mr. Morgan next insinuates, in page 54, that Mr. Babbage treats the decennial investigations which are submitted to the members as a fallacy calculated only to delude the public. This passage also, like the former, exists only in Mr. Morgan's imagination, since there is no expression in Mr. Babbage's work tending in the most remote manner to such an opinion; and it can only be by a singular perversion of the ordinary meaning of words that such an interpretation could have been deduced from anything that he has said.

Mr. Morgan further states, in page 56, that Mr. Babbage accuses the Society of injustice, in the unequal method of distributing the surplus profits of the Society. Now, it is true that Mr. Babbage has expressed an opinion (not amounting, however, to an accusation) that the mode adopted by the Equitable Society is not the most equitable mode of distributing the surplus

profits of an Assurance Society, and in this opinion he is supported by many others as conversant with the subject as Mr. Morgan is; but Mr. Morgan ought in candour to have quoted the whole paragragh, which would have removed the sting that, under the colour of his misrepresentation, appears to exist. This formidable passage, which seems to have given so much offence, is in the following words:—"This inequitable mode of apportioning the profits is not, however, so disadvantageous in a system of mutual assurers, because the reserved surplus again accumulates for the benefit of the assured at the next period of division." In Mr. Morgan's opinion, it seems little short of high treason even to doubt the wisdom of any of the rules and regulations passed at Chatham Place.

Mr. Morgan, in page 59, represents Mr. Babbage as recommending that the division of profit should be made annually, and the whole of the surplus distributed amongst the members; but this is not the fact. Mr. Babbage certainly leans to the opinion that the divisions of the profits of an Assurance Office (not alluding, however, to the Equitable) ought to be more frequent; and I perfectly agree with him that, by the aid of an improved system of keeping accounts, it would by no means be difficult to assign annually to each assurer his proportion of the profits. But Mr. Babbage nowhere states that the whole of the profits onght to be so distributed; and these two opposite and distinct subjects ought not to have been thus unnecessarily united and placed in juxtaposition, in order to raise a prejudice against Mr. Babbage's opinion on these subjects.

Mr. Morgan further accuses Mr. Babbage, in page 61, of inaccuracy, in stating that no benefit attaches to an assurance (subsequent to its becoming one of the fortunate 5,000) until the next succeeding order; and, with an air of triumph, he announces that it immediately comes within the order for prospective additions, as soon as it falls within the golden number above mentioned. But Mr. Morgan must have well known that Mr. Babbage was speaking of the retrospective, and not of the prospective, additions—a distinction better known in its practical application than in the obscure wording of the rules of the Society. In this point of view Mr. Babbage was correct. The former term alludes to the large decennial divisions of profit, where the *bonus* is multiplied by the number of years that the assurer has been in the Society, whereas the other addition is very small in amount, and has not even been particularly alluded to by Mr. Babbage.

But I shall not proceed further in these remarks. I have already said enough to show the spirit in which Mr. Morgan's pamphlet is written, and how unfairly he has treated Mr. Babbage, both in his quotations and in his comments. The high character and splendid abilities of these two gentlemen ought to have removed all ground for even the suspicion of misrepresentation. Mr. Babbage's work was written with a view to point out the various shades of difference between the several Offices established in London for the assurance of lives, as well as to furnish some useful hints to those who might be disposed at any future time to establish new ones. In doing this he has certainly rendered an essential service to the public; and the Equitable Society, so far from being censured, as insisted upon by Mr. Morgan, is everywhere held up (as it deservedly is) as the best institution of the kind in existence.

Mr. Morgan seems very much annoyed at the interest which the public take in the proceedings at Chatham Place; and "cannot see what the public, distinct from its members, has to do with the concerns of the Society." But

Mr. Morgan ought to know, that when a Society issues its proposals to the public, thus inviting them to join their concern, the public have a right to canvass and discuss the proposals thus tendered to them, and that in such discussion the whole establishment comes under review. Besides, where so many millions of money are concerned, it is not the members alone that are interested, since the prospect of ultimately enjoying this property is diffused, not amongst the members themselves, but amongst a vast assemblage of relations and expectants, all of whom consider themselves as fairly entitled to give an opinion on what is going forward.

The Equitable Society, however, need not fear the test of any public examination or inquiry. For, though its origin, according to Mr. Morgan's own showing, was not very flattering, and although in its infancy it seems to have adopted practices which are deprecated in even the worst of the present Joint Stock Companies, yet, by a speedy reformation, the Society advanced in credit and renown; and, under the prudent and able administration of Mr. Morgan himself, it has arrived at a degree of opulence and importance unparalleled in the history of such undertakings.

I have the honour to be, Sir,

Your obedient servant,

FRANCIS BAILY.

37, *Tavistock Place*, *June* 24, 1828.

To the Editor of the Times.

Sir,—I feel no disposition to enter into any dispute or discussion respecting my late publication, and therefore have only to observe, in answer to Mr. Baily's letter in your paper, that he has put a construction on the remarks I made on Mr. Babbage's work which I never intended. In using the word misrepresentation, I had too much respect for that gentleman to suppose that it could have arisen from any other cause than misinformation, and have accordingly regretted that he should not have been better acquainted with the rules and practice of the Equitable Society before he made them the subject of his animadversion. With regard to the passages in which I am accused of having misrepresented him, I see no reason to alter my opinion, and consequently do not think it necessary to notice all the instances in which I am charged with acting unfairly towards Mr. Babbage. I shall just select the first and second articles in Mr. Baily's letter, which will be sufficient for my present purpose, and must refer to the works themselves for a justification of the rest.

In the first article it is said "that I commenced my attack on Mr. Babbage by accusing him of stating that the Equitable Society had a large unfunded capital, which they concealed from the knowledge of the public." Mr. Babbage's words are—"It is said that they possess a large undivided capital, of which the public, from want of sufficient information, have very vague, and, perhaps, extravagant ideas" (page 40). In giving this account, without expressing a doubt as to its authenticity, Mr. Babbage so far sanctions it, and I was naturally led to express my surprise from what source he could have derived his information, and how he could have given any credit to such a tale.

In the second article it is said "that I have accused Mr. Babbage with treating the decennial investigations which are submitted to the members as a fallacy calculated only to delude the public." Now this is not strictly

correct. I have only observed, that, from the manner in which these statements are represented to have been given, it might be inferred that the whole was a fallacy, calculated only to delude the public; and I wish to know what other inference could be deduced from the three following passages:—"Although two-thirds is the sum nominally divided amongst the assured, since that sum is only payable on the death of the respective nominees, it is not in reality equal to two-thirds, and is, in fact, different for different ages" (page 81). Again, "these facts show that the two-thirds apparently given to the assured, are, in reality, not equal to one-half the total profits" (page 82). And, in page 83, "that two-thirds of the profits are not divided among the assured, as they are led to believe in the prospectus." With such an accusation against them, is it possible to think respectfully of the abilities or integrity of those who conduct the affairs of the Society? Mr. Baily mentions his having read Mr. Babbage's work before its publication, and therefore makes himself more blameable on this occasion than Mr. Babbage, who must have naturally concluded himself to be correct when supported by a gentleman, not only so well acquainted with the subject, but also who had been so many years a member of the Society.

Had Mr. Babbage applied to myself before, as he did after, the publication of his work, I have no doubt that he would have corrected most of the mistakes which I pointed out to him, and I should have been spared the unpleasant task of noticing them in public, which has been really done, not with the view of "passing unmerited censures upon him," but of rectifying his mistakes, and thus preventing the Society from being misled by them.

I am, Sir, yours, &c.,

Equitable Assurance, June 30, 1828. WM. MORGAN.

On the proper Mode of estimating the Liabilities of Life Insurance Companies. By ROBERT TUCKER, *Actuary to the Pelican Life Insurance Company, and one of the Vice-Presidents of the Institute of Actuaries.*

[Read before the Institute, 24th November, 1862, and printed by order of the Council.]

THE object of this paper is to endeavour to show that the true method of estimating the liabilities of an Insurance Company should be based upon the principle of re-insurance by its own rates of premium, interest being reckoned at 3 per cent.

In submitting the following observations, I feel it necessary, in the first place, to apologise for recurring to a subject which has been so frequently and ably treated by the President of the Institute of Actuaries. But as my method of dealing with this important branch of our business differs in some respects from the plan laid down by that learned writer, I trust I may be allowed to record the present attempt to demonstrate what in my opinion are

the true principles which should guide us in determining the liabilities of Life Insurance Companies.

Let π'_x denote the Office annual premium at any age x,
a'_x the corresponding life annuity, and
A'_x the corresponding single premium;

then the liability of a Company in respect of each policy at the end of the first period of division, say n years, may be determined from any one of the following well-known formulæ, viz.:—

$$(\pi'_{x+n}-\pi'_x)\cdot(1+a'_{x+n});$$

or,
$$A'_{x+n}-\pi'_x\cdot(1+a'_{x+n});$$

or,
$$1-\frac{1+a'_{x+n}}{1+a'_x};$$

or,
$$1-\frac{\pi'_x+d}{\pi'_{x+n}+d};$$

If π_x denote the pure annual premium, and k the percentage of loading, then

$$\pi'_x=(1+k)\pi_x.$$

Suppose an Office to charge premiums based upon the Carlisle 3 per cent. table, with a loading or addition to each premium of 25 per cent., then we have the following values:—

Age.	π'_x.	a'_x.	A'_x.
20	1·8667	19·9223	39·0613
30	2·4400	17·6828	45·5841
40	3·2491	15·2291	52·7202
50	4·5279	12·4398	60·8546

&c. &c.

The value of a policy of £100 at several decennial periods will be as follows:—

Age when Assured.	Period Elapsed.				
	10 Years.	20 Years.	30 Years.	40 Years.	50 Years.
20	10·7039	22·4315	35·7631	52·9084	67·3967
30	13·1334	28·0630	47·2635	63·4885	76·2871
40	17·1869	39·2903	57·9684	72·7020	82·3946
50	26·6907	49·2452	67·0366	78·7408	
60	30·7662	55·0351	71·0007		

The value of a policy at the same ages and periods, when the loading is thrown off and the pure premium only is taken into account on both sides, is

Age when Assured.	Period Elapsed.				
	10 Years.	20 Years.	30 Years.	40 Years.	50 Years.
20	9·4219	20·0609	32·5747	49·3662	64·2066
30	11·7455	25·5617	44·1001	60·4839	73·9026
40	15·6549	36·6598	55·2255	70·4304	80·7142
50	24·9041	46·9138	64·9412	77·1365	
60	29·3103	53·3151	69·5537		

On comparing these two sets of values, it is noticed that the reserve, when the *pure premium* only enters both sides of the account, is less than when the estimate is made by the *full premium.*

The reason of this difference will be apparent when we consider the effect of a pure premium valuation upon old and new members. A member now aged 60, but who entered at 20, pays, in addition to the net value of the risk, 7*s*. 6*d*. per cent. per annum, whilst a member entering at age 60 pays £1. 9*s*.

By the pure premium method of valuation the contribution of a member at age $(x+n)$ years is—the original premium,

$$\pi'_x \text{ or } (1+k)\pi_x;$$

and the reserve,

$$\pi_{x+n}-\pi_x;$$

together,

$$=\pi_{x+n}+k\pi_x.$$

By the other method, the contribution is—the original premium,

$$\pi'_x;$$

and the reserve,

$$\pi'_{x+n}-\pi'_x;$$

that is,

$$\pi'_{x+n}=\pi_{x+n}+k\pi_{x+n};$$

If the loading is a constant quantity, $k\pi_{x+n}=k\pi_x$, and the inequality above noticed would be corrected. But an uniform addition to the premium is objectionable in practice, because it makes the rates too high at the younger and too low at the older ages. Besides the reason assigned for this plan—viz., that each person should pay the same contribution, is true only as to amount, and not in respect of value. For A'_x-A_x will decrease as the age increases when the loading is constant, but this is not so when it

is by a percentage on the premium, as will be seen in the following examples deduced from the Carlisle 3 per cent. table—A'_x denoting the value of the annual premium + 25 per cent., A_x the pure premium :—

	Age 20.	30.	40.	50.	60.
A'_x	39·0613	45·5841	52·7202	60·8546	71·3030
A_x	33·8973	40·1254	47·1580	55·4303	66·5299
	5·164	5·4587	5·5622	5·4243	4·7731

The custom of paying an annual commission of 5 per cent. on a premium would seem to offer another practical objection to a constant loading. But this difficulty is sought to be overcome by making the loading to partake of both characters—a constant and a percentage. In whatever way the annual premiums are loaded, if they are not made the basis for calculating the other rates of the Office, it must lead to some anomalies in its premiums.

The following rates appear in the prospectus of an Office having a consulting actuary :—

Survivorship Rates.					*Joint Life Rates.*				
20 *v.* 25=	£1	5	10	per cent. per annum.					
25 *v.* 20=	1	10	9		20 and 25=	£3	13	1	per cent. per annum.
	£2	16	7						
40 *v.* 45=	£2	0	1						
45 *v.* 50=	2	14	8		40 and 45=	£5	13	0	
	£4	14	9						
29 *v.* 20=	£1	6	11						
			2		20 and 20=	£3	9	10	
	£2	13	10						

And, generally, throughout the table similar irregularities are to be met with. So that, in all cases whenever it is desired to make a joint life insurance, a considerable saving may be effected by taking out two survivorship policies.

It is difficult to discover upon what principle these tables have been constructed—certainly without regard to the equation

$$\frac{A_{x,y}}{1+a_{x,y}} = \frac{A^{1}_{\overline{x},y} + A_{x,\overline{y}}^{\;1}}{1+a_{x,y}}.$$

Indeed, it is obvious that, unless the relation between these and other algebraical values of different descriptions of risk is to be maintained, the elements employed must be those which are immediately deducible from the ordinary rates of premium.

For instance, if $\pi'_x = 1\cdot8667$ at age 20,

$$a'_x \text{ must} = 19\cdot9223,$$
$$\text{and } A'_x \quad = 39\cdot0613,$$

when interest is reckoned at 3 per cent.

Now, suppose an Office, as some have been known to do, to base their annual premiums upon the Carlisle 4 per cent. table, with a loading of 40 per cent., we should have, at age 75,

$$\pi_x = \cdot12182 \qquad a_x = 5\cdot239 \qquad A_x = \cdot76004$$
$$\pi'_x = \cdot17055 \qquad a'_x = 3\cdot785 \qquad A'_x = \cdot816$$

But if the custom had been to add 40 per cent. to all the Office premiums, the single premium at age 75 would be $1\cdot06406$; that is, an Office would require £106. 8*s.* 2*d.* to insure £100 at death, which is absurd.

A'_x being the premium to insure £1 at death, its value can never be made to appear greater than unity, and must, therefore, be determined from $\pi'_x.(1+a'_x)$, and not $\pi'_x(1+a_x)$.

In constructing a table of annual premiums it is not material whether the rate of interest employed is 3, 3½, or 4 per cent., provided the percentage added to each premium increases with the rate of interest. For instance, if we take the Carlisle table, it will be found that 3 per cent. premiums loaded with 25 per cent., or 3½ per cent. with 30 per cent., or 4 per cent. with 35 per cent. added to each premium, are nearly equal at all the important ages. It is only at the very early periods of life that the rates are lower, and at the more advanced ages higher, than we are accustomed to see them when 4 per cent. interest is used.

But it is otherwise when we come to estimate the liabilities of a Company, for then the higher the rate of interest used the smaller will be the reserve. It would therefore appear, that practically and for the sake of uniformity, as well as for safety, it is not desirable to use a higher rate of interest than 3 per cent. in the construction of assurance premiums, and in valuing the liabilities of any Company using them.

If we recognise the principle that from each person effecting an insurance for the whole term of life the Company should derive the same profit upon each £100 assured, the loading must be by a percentage on the annual premium; so that $A'_x - A_x$ may be constant, or nearly so, at all insurable ages. It should be borne

in mind that the entrant is for life; that the value of the risk is determined in a single payment, and that the annual contribution is merely its equivalent spread over the duration of the risk.

We have seen that a *pure premium* valuation not only gives a smaller reserve than a valuation by the *full premium*, but it is unfair to incoming members, because they are charged a higher rate of contribution than those already admitted. When the business of one Office is transferred to another, they are said to amalgamate;* and if, after a valuation made upon the same principles by both Offices, one is found to be richer than the other, the surplus in the one case is got rid of, or the deficiency in the other is made good, in order that they may amalgamate upon equal terms. The principle of equality is here clearly recognised, but it is not observed as between old and new members in an Office valuing its pure premiums only.

The following example is given at page 186 of vol. iii. of the *Magazine*:—It is supposed that "a Society charging the Northampton premiums, and making its calculations by that table, to have been in existence 40 years, and to consist of 4,000 members, each assured for £1,000, at the age of 30; but the whole body divided into four classes of 1,000 each, the duration of whose assurances is 10, 20, 30 and 40 years respectively. It is further supposed that the real rate of interest prevailing is 3 per cent., and the real mortality that called the 'experience.'"

To the liabilities thus deduced I will add those derived from the Carlisle 3 per cent. pure premiums and the Carlisle 3 per cent. premiums with a loading of 25 per cent.

	Northampton Table.	Experience.	Carlisle, Pure.	Carlisle, Loaded.
	£	£	£	£
Value of £1,000,000 assured 10 years†	141,978	145,947	136,974	155,734
" " " 20 "	276,590	305,089	275,136	305,030
" " " 30 "	425,051	480,059	460,520	497,035
" " " 40 "	594,938	648,611	624,358	659,285
	1,438,557	1,579,706	1,496,988	1,617,084

* See Mr. Jellicoe's paper "On the Principles which should govern Assurance Companies in Amalgamating," 28th Feb. 1858.

† The values given in Mr. Jellicoe's examples suppose the year's premium to have been just paid: I have, for the sake of comparison, made the same assumption in the other cases.

It will be observed that the valuation by the Carlisle loaded premiums gives a larger reserve than is shown by any of the other rates; that it is only about $2\frac{1}{3}$ per cent. in excess of that recommended by Mr. Jellicoe, but is 8 per cent. higher than by the Carlisle pure premiums, and nearly $12\frac{1}{2}$ per cent. by the Northampton Table.

We are warned that the "good old Northampton table" is rather a dangerous one to trust to. Undoubtedly it is, as Mr. Jellicoe proves by the example just quoted. But that "a similar result may be looked for whenever the rates of interest are the same and the true mortality less than that of the factitious table," is not quite so apparent. I have shown practically in the examples already given, and I shall presently demonstrate theoretically, that the reserve by a table of premiums loaded with a percentage is greater than the reserve by a table of pure premiums.

But suppose we value by what is called a *true table* and at a *true rate of interest*, and deduct from the value of the future premiums a percentage for profit and expenses: if the annual premiums, or some of them, have been commuted for a single payment, the plan fails, and the reserve becomes insufficient unless some contrivance is resorted to to give a greater margin. Besides the deduction, if arbitrary, is objectionable; and if it is the exact value of the loading, the reserve is the same as by the pure premium method. For

$$A_{x+n}-\pi'_x(1+a_{x+n})-k\pi_x(1+a_{x+n})=A_{x+n}-\pi_x(1+a_{x+n}).$$

The effect of any mode of valuation is to assume that certain premiums will be payable in future upon the policies existing and included in such valuation. And if an Office distributes its surplus according to this estimate, it places itself, in point of security, in the same position as if those premiums had been originally charged. Because, as the late Mr. David Jones remarks at page 1094 of his treatise *On the Value of Annuities and Reversionary Payments*, the sum reserved and the value of the future premiums are together equal to the single premium that would be charged to reassure the various lives at their present ages according to the particular table used for the valuation.

Suppose an Office to anticipate the value of the loading upon all its policies, then we should have, for the value of the future premiums,

$$\pi'_x.(1+a_{x+n});$$

and for the reserve or value of the policies,

$$(\pi_{x+n}-\pi'_x).(1+a_{x+n});$$

these together,

$$=\pi_{x+n}(1+a_{x+n}) \text{ or } A_{x+n},$$

the single premium required to reinsure all the lives at their present ages, without any margin for expenses, commission, &c.

If the valuation were made by the Carlisle 4 per cent. table, an

Office charging originally Carlisle 3 per cent. premiums, with 25 per cent. added thereto, would have in hand and in the value of the future premiums upon every £100 insured—

At age 20 . .	25·532	instead of	39·061
30 . .	31·338	"	45·584
40 . .	38·178	"	52·720
50 . .	46·658	"	60·855
60 . .	58·987	"	71·303

Let V'_{x+n} represent the value of a policy by Office premiums at age $(x+n)$ years,

V_{x+n} the value by pure premiums at same age,

then
$$V'_{x+n}=(1+a'_{x+n}).(\pi'_{x+n}-\pi'_x).$$

Now,

$$1+a'_{x+n}=\frac{1}{\pi'_{x+n}+d}=\frac{1}{(1+k)\pi_{x+n}+d}$$
$$=\frac{1}{\dfrac{1}{1+a_{x+n}}+k\pi_{x+n}}=\frac{1+a_{x+n}}{1+k\pi_{x+n}(1+a_{x+n})},$$

and

$$(\pi'_{x+n}-\pi'_x)=(1+k)(\pi_{x+n}-\pi_x);$$

$$\therefore V'_{x+n}=\frac{1+k}{1+k\pi_{x+n}(1+a_{x+n})}\times(1+a_{x+n}).(\pi_{x+n}-\pi_x)$$
$$=\frac{1+k}{1+k\pi_{x+n}.(1+a_{x+n})}\times V_{x+n}.$$

If, instead of an Office debiting and crediting itself with the value of the pure premium or the full premium, it debits itself with value of the pure premium and takes credit for the value of the full premium, then its liability is represented by

$$(\pi_{x+n}-\pi'_x).(1+a_{x+n}), \text{ or}$$
$$\{\pi_{x+n}-(1+k)\pi_x\}(1+a_{x+n}), \text{ or}$$
$$(\pi_{x+n}-\pi_x)(1+a_{x+n})-k\pi_x(1+a_{x+n})=V''_{x+n} \text{ suppose,}$$

$$\text{then } V''_{x+n}=V_{x+n}-k\pi_x(1+a_{x+n});$$

and, since

$$V'_{x+n}=V_{x+n}\times\frac{1+k}{1+k\pi_{x_{x+n}}(1+a_{x+n})}$$

$$V''_{x+n}=V'_{x+n}\times\frac{1+k\pi_{x+n}(1+a_{x+n})}{1+k}-k\pi_x.(1+a_{x+n}).$$

These expressions serve to point out the difference in the value of a policy by the three methods here employed.

z 2

$$V_{x+n} \text{ is} < V'_{x+n} \text{ by the quantity } \frac{1+k}{1+k\pi_x(1+a_{x+n})},$$

$$\text{and } V''_{x+n} \text{ is } < V_{x+n} \text{ by the quantity } k\pi_x(1+a_{x+n});$$

showing, in this last case, that the remaining value of the original loading is anticipated. They also enable us to pass from one value to another; but nothing is saved in point of labour by doing so, because it is quite as easy to calculate the value of a policy by each method separately and compare the result.

To find the relation between A'_x and A_x, and between $1+a'_x$ and $1+a_x$:—

$$1+a_x = \frac{1}{\pi_x+d},$$

$$1+a'_x = \frac{1}{\pi'_x+d} = \frac{1}{(1+k)\pi_x+d} = \frac{1}{\frac{1}{1+a_x}+k\pi_x};$$

$$A_x = \pi_x.(1+a_x)$$

$$\begin{aligned} A'_x &= \pi'_x(1+a'_x) \\ &= (1+k)\pi_x.(1+a'_x) \\ &= \frac{(1+k)\pi_x}{\frac{1}{1+a_x}+k.\pi_x} \\ &= \frac{1+k}{\frac{1}{\pi_x(1+a_x)}+k} \\ &= \frac{1+k}{\frac{1}{A_x}+k}. \end{aligned}$$

I think I have now shown that the proper mode of estimating the liabilities of Life Insurance Companies is by the principle of reinsurance. This principle is clearly recognised and acted upon by Fire and Marine Insurance Companies. In determining what profit these latter Companies have made, it is their practice to set aside a sufficient sum to reassure their outstanding risk. I mean such a sum as another Office charging similar premiums would be willing to undertake the liabilities for, and not an arbitrary sum which some one or more Offices desirous to extend their operations might be content to take.

This principle is also recognised in the standard works on life contingencies, and acted upon for many years by all actuaries.

When an Office values its liabilities upon the principle of reinsurance, no doubt, I submit, can arise as to its position at any given rest. For the value of the depreciation by reason of the increased age of the lives assured—that is, the value of the difference between the premiums that would be required to reinsure the amount at risk and the premium actually receivable—is determined; and this can only be ascertained by this particular mode of valuation. If the funds of the Company other than the paid-up capital or what belongs exclusively to the proprietors, exceeds the value of the outstanding risk, the *difference* is the *surplus* for division. If the assets fall short of this value, the *difference* is the *deficiency*, which ought to be made good out of the proprietors' fund. It is necessary, of course, in the first instance, to charge such premiums as may fairly be taken to represent the true value of the risk, and to add a percentage thereto sufficient for expenses and bonus purposes; otherwise inadequate rates of premium will lead to insufficient reserves by this as well as by the pure premium method. The complaint that is frequently made against the mode of valuation here contended for is that we are working in the dark, and we know not what margin is set aside at each periodical rest. This complaint has no foundation in fact—the method is the most simple that can be devised—the margin reserved being the loading at the present age of each assured life, instead of the original loading.

Suppose £1,000,000 to be now assured on lives all aged 60 years, the policies to have been in force 30 years, and the additions made by way of bonus to be £200,000. The value of this risk by the Carlisle loaded table is £615,241, and by the Carlisle pure table £574,061. In the former case the Office reserves the loading on £1,200,000, at age 60, which is £1. 9*s*. per cent., and amounts to £17,400 per annum. In the latter the reserve is the original loading on £1,000,000, at age 30, and only amounts, at 9*s*. 9*d*. per cent., to £4,875 per annum. I admit that it has its inconveniences, because a bonus can only be declared out of realised profits, and that it is not possible to draw upon futurity without showing what amount has been anticipated.

On the other hand, the practice of treating as an asset the value of what has been termed "the marginal guaranty," and which has yet to be realised, affords an opportunity of declaring profits out of capital which has not been neglected. Indeed instances are on record where the whole of this "marginal guaranty" has been anticipated.

This subject has of late years engaged the attention of many

persons interested in the welfare of Life Assurance Institutions. The aid of actuaries has been sought to test the solvency of a particular Company, or to ascertain whether the members of a Society receive their due share of profit. It is much to be regretted that these questions do not always receive a satisfactory solution, owing, it is believed, to the difference of opinion which is said to prevail amongst actuaries of more or less eminence in their profession. But is this question a mere matter of opinion? I think not. When an actuary is asked, How should an Office using a given table value its liabilities, so that no portion of its future profits may be anticipated, and that old and new members of the same age may stand upon an equality, there can be but one answer—Upon the principle of reinsurance here laid down.

On the Valuation of Policies of Assurance.

(Concluded from page 267.)

CASE 6.—*Surrender value, or the price the holder of a policy should receive from the Office on a mutual cancelling of the engagement.*

THE premium paid on a policy of assurance is divisible into two parts—the first, or net premium, representing the annual contribution absolutely necessary, on a calculation of averages, to provide a stipulated sum on death; and the second being an arranged contribution towards the expenses of the Office.

The contract entered into by the parties to the policy, is—

> On the part of the assured, to pay a certain sum annually till his death.
>
> On the part of the Office, to pay to the representatives of the assured, on his death, or at a reasonable time afterwards, the sum assured, on condition that all the agreed payments on the part of the assured have been duly made.

If the assured wish to break his part of the contract, *i.e.*, to withdraw those contributions on the expectation of the continuance of which the calculations of the Office are based—he would have no reasonable ground of complaint if forfeiture were the result; but it is customary with Offices to make an allowance in consideration of the surrender, on the ground that the payments already made have been more than sufficient to meet the risk already run.

Rule B would give the surrender value of the policy, on the following conditions, viz.:—

1st. That the premiums charged were absolutely net premiums, or increased by an equal charge per £1 upon the sum assured in order to meet expenses, and that the annuity value was also a net value.

2ndly. That the experience of the Office exactly coincided with the data used in the calculation of its tables.

3rdly. That no injury results to the Office or to other contributors by the withdrawal of the assured.

The first of these conditions is rarely, if ever, fulfilled; the second has never been; and the third is a question that can be decided only on a scrutiny of the Office accounts.

The real value of a policy, upon its surrender to the Office, depends on (1st) the mortality experienced by the Office, (2ndly) the rate of interest made on its investments, (3rdly) the proportion borne by its actual expenses to those which have been assumed in the preparation of its tables, and (4thly) on the constitution of the Society.

In a purely Mutual Society, there can be no doubt that each policy-holder is entitled to a share of any profits that may accrue from either a favourable rate of mortality, a more profitable rate of investment than that anticipated, or an expenditure less than that which might have been incurred in strict accordance with the premiums charged; and, on the other hand, there can be no doubt that he should bear his fair proportion of all losses resulting from the operation of contrary causes.

The actuary of the Office can alone procure, from the recorded transactions of the Society, the necessary data for the correct determination of the surrender value of a policy; and if he, on investigating the affairs of the Society, determine these, the problem may be solved thus:—

Ascertain the net value of an annuity due on the life at the present age, at the experienced rate of mortality and interest; multiply it by the net premium that would be chargeable at the present age, less the gross premium chargeable, on the same data, at the age when the policy was taken up; and add to, or deduct, from the result, the amount, in as many years as the policy has been in force, of an annuity of the difference between the gross premium that should have been charged and that which has been paid, according to the circumstances of the case. [*Rule* M.]

This Rule may be elucidated thus:—

The assured pays, at the time of settlement—

1st. The sum to which the adjusted gross premium for the younger age would have amounted at the time;

2ndly. The present value of all future payments of the same premium;

and, completing thus his part of the contract and entitling himself to the full benefit of the policy, receives—

1st. The present worth of the sum assured;
2ndly. A return of all the payments he has already made.

The surrender value of the policy before referred to, if the Office experience coincide with the English Life Table at 3 per cent. and the expenses are found to be fairly adjusted by an addition of £3 to the net premium at age 20, would be found thus:—

Net premium at age 60		£61·312
Gross premium at age 20		18·851
Difference . . .		42·461
Mutiply by worth of annuity due, at age 60 .		11·057
		469·49
Gross premium at 20 . .	£18·851	
Premium paid	17·290	
	1·561	
Multiply by amount of annuity in 40 years	75·4013	
Deduct the product, the premium paid having been insufficient		117·70
		£351·79

If, however, the Office make 4 per cent. interest, and an addition of £2. 10*s.* to the premium would have sufficed for expenses—

Net premium at age 60		£58·67
Gross premium at age 20		16·60
		42·07
Multiply by worth of annuity due at age 60 . .		10·295
		433·11
Gross premium at 20	£16·60	
Premium paid	17·29	
Excess payment . .	·69	
Multiply by amount of annuity in 40 years	95·026	
Add the product, the premium paid having been too large		65·568
		£498·678

If the Office be a Proprietary or a Mixed Office, the non-participating assured, by selecting a mode of assurance that deprives him of the right to a participation in profits, screens himself also against liability as regards losses; in such cases it appears only right that, on cancelling the contract, the consideration-money should be merely the difference between the net present value to the Office of the sum assured which the policy-holder surrenders and the net present value of the future premiums surrendered by the Office. The Rule will then be expressed thus:—

Add to the premium payable under the terms of the policy the discount for one year on the sum assured (reckoned at the rate of interest actually made by the Office), multiply by the net value of an annuity due on the surrenderer's life, and deduct the product from the gross sum assured. [*Rule* N.]

In the case before used, the valuation would be, if the Office is making 4 per cent. of its funds—

Premium	£17·29	
Discount on £1,000 at 4 per cent. . .	38·46	
	55·75	
Annuity due, at age 60, worth, by same table as in last example . . .	10·295	
Deducting the product	573·946	from £1,000

The surrender value, on these terms, is £426. 1*s.* 1*d.*

If the Office be making only 3 per cent., the valuation would be—

Premium	£17·29
Discount at 3 per cent.	29·126
	46·416
Annuity due, worth	11·057
Deducting the product	£513·22 from £1,000

The surrender value is found to be £486. 15*s*. 7*d*.

A Method of Multiplication which may be practised Mentally. By LIEUT.-COL. WM. HENRY OAKES, *Bengal Presidency.*

1. WRITE down the multiplicand in the usual manner.

2. Write on a separate slip the multiplier with its figures reversed (as in contracted multiplication), being careful to space the figures so that they can be exactly placed over those of the multiplicand.

3. Thus arranged, place the multiplier so that the figure in the true unit's place may fall over the unit's place of the multiplicand; then multiply each figure of the multiplicand by that immediately over it, sum the products and write down the last figure of the sum, reserving the rest as a carriage.

4. Shift the multiplier one place to the left, multiply each figure as before and sum the products, adding thereto the carriage from the preceding operation; write down the last figure of this new sum, and reserve the rest as a carriage.

5. Proceed in this manner until the multiplier is exhausted; the sum last obtained must be written in full, and the work will then be complete.

Example.—$\overset{a}{9}\overset{b}{7}\overset{c}{6}\overset{d}{3} \times \overset{l}{8}\overset{m}{4}\overset{n}{5}\overset{r}{2}$.

Process in detail, showing the several positions of the multiplier.

```
      2548 Multiplier reversed.
   9763    Multiplicand.
   ----
      6
```

	2548
	9763
	15
	12
	27
	2548
	9763
	12
	30
	14
Carriage . . .	2
	58
	2548
	9763
	24
	24
	35
	18
Carriage . . .	5
	106
	2548
	9763
	48
	28
	45
Carriage . . .	10
	131
	2548
	9763
	56
	36
Carriage . . .	13
	105
	2548
	9763
	72
Carriage . .	10
	82

Mental process.

6	$= rd.$
1 5	$= nd.$
1 2	$= rc.$
2 7	sum.
1 2	$= md.$
3 0	$= nc.$
1 4	$= rb.$
5 8	sum.
2 4	$= ld.$
2 4	$= mc.$
3 5	$= nb.$
1 8	$= ra.$
1 0 6	sum.
4 8	$= lc.$
2 8	$= mb.$
4 5	$= na.$
1 3 1	sum.
5 6	$= lb.$
3 6	$= ma.$
1 0 5	sum.
7 2	$= la.$
8 2 5 1 6 8 7 6	

The figures of the *result only* are to be recorded, as shown by the dotted lines.

On the Methods pursued in Valuing the Risks of Life Assurance Companies, and on the Division of Surplus. *By* Charles Jellicoe, *Actuary of the Eagle Insurance Company.**

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In a paper read a short time ago before the Institute, I endeavoured to show the inexpediency of using, for the purposes of valuation, tables constructed on the principle which characterizes the Northampton table. I did not intend to pursue this particular branch of the subject further; but, finding that more difference of opinion exists in reference to it than I was aware of, and considering that the practice of making valuations by the Northampton table, and by tables framed upon similar principles, is still carried on to a great extent, I am led to offer a few more observations upon the matter previously to entering upon the consideration of some other questions, the correct solution of which materially depends upon a right understanding of this one.

In any table of premiums intended for actual use, it is evident that the rates charged must be more than sufficient to provide for the sum assured. This is obviously demanded by prudential considerations, and is, indeed, based upon an inevitable necessity; but in order to judge of the sufficiency of the premium to be charged, it is plain that we must know the amount of that which is required to provide for the risk merely, the determination of the one being essential to the proper adjustment of the other. Now, the premium for the risk, or the payment required to provide for the sum assured, can only be deduced, as is well known, by means of a mathematical process, from the true average rates (or from what are supposed to be the true average rates) of interest and mortality, whilst the addition to be made to the payment thus deduced, as a provision for other contingencies, is purely arbitrary. In this view of the matter, then, the test of the accuracy of the data will be their yielding a premium just sufficient, and no more than sufficient, to provide for the sum assured. It will be convenient to call such data, or such rates of interest and mortality, the true data or rates; and such premium, the true premium. Now, if the premiums to be charged be derived, by a direct process, from

* This and another paper on the same subject by the same writer are, we believe, the only ones published by authority of the Institute not in the *Journal*. For the sake of completeness, we avail ourselves of a convenient opportunity to reprint, with some slight additions, the longer one of the two.—Ed. *A. M.*

rates of mortality and interest, it is evident that such rates cannot both be true ones, since they yield a premium more than sufficient for the sum assured; that is to say, they yield a false premium, and therefore must, one or the other, or both, be also false.

I will call tables of this description "factitious," *as being made up of quantities in their nature heterogeneous*, and as distinguished from tables having to do solely with the premium for the risk. Of these factitious tables, many, beside the Northampton, are extant, although not published; for it is well known to be a common practice to arrange, first, the premiums to be charged; then to obtain the values of the annuity from the premiums; and from the values of the annuity to deduce rates of interest and mortality; such values and rates being thereafter made use of in all official calculations. But although the distinguishing feature of tables thus constructed is strictly identical with that which characterises the Northampton table, its introduction in the case of the latter was owing to circumstances wholly different, and was, indeed, scarcely to be avoided by the original framers of the table, as I may presently have occasion to show; but I would first endeavour to point out more distinctly the relation which these factitious tables bear to the true ones; or, what is the same thing, the relation which exists between the elements of a table, when such elements include the provision for extra contingencies, and when they do not, and to call attention to some of the consequences resulting from their use. With this object, let us call p'_x the true premium—that is to say, the premium for the risk; and let d' denote the discount of £1 at the true rate of interest, then the expression for the corresponding value of the annuity will be

$$\frac{1}{p'_x+d'}-1=\mathrm{A}'_x.$$

In like manner the annuity answering to p_x, or the premium charged, may be expressed by

$$\frac{1}{p_x+d}-1=\mathrm{A}_x,$$

where p_x may be considered as equal to $p'_x+\phi_x$, and d taken at any rate of interest whatever. This, then, is the relation between the value of the annuity and the premium charged, which obtains in the Northampton table, and of course in all other tables similarly constructed.

Substituting, in the latter of the two expressions, the value of p'_x, as shown by the former, it will be found that

$$1+\mathrm{A}_x = \frac{1+\mathrm{A}'_x}{1+(\phi_x+d-d')(1+\mathrm{A}'_x)},$$

an equation which exhibits the relation between the values of the true and the factitious annuity, or between the value of the annuity when the payment for extra contingencies is involved in it, and when such payment is not so involved.

Now, the mere inspection of this quantity ought to suffice, one would think, to show the entanglement and obscurity which must result from its introduction into our investigations: but it is worth while to examine the manner in which it operates a little more closely. Between the limits within which our practice is usually confined, the difference between d' and d can rarely exceed ·00933, that being their difference when taken at two consecutive and integral rates of interest; but whatever their difference may be, ϕ_x will generally be found in the cases which occur in practice to exceed it; hence (ϕ_x+d-d') is almost always positive, and, consequently, $1+\mathrm{A}_x$ is almost always less than $1+\mathrm{A}'_x$. The effect of this upon other quantities depending on the value of the annuity is obvious enough. Thus the value of a reversion by the factitious table will generally be greater than by the true one, since $1-d(1+\mathrm{A}_x)$ evidently exceeds $1-d'(1+\mathrm{A}'_x)$ (d being equal to or less than d'); and conversely, the present values being the same, the reversionary sums will of course be less. Moreover, the liability under an assurance will generally be less by the factitious table than by the true one, especially if ϕ_x be a constant or decreasing quantity, for in such cases

$$(p'_{x+x}+\phi_{x+n}-p'_x-\phi_x).(1+\mathrm{A}_{x+n})$$

is clearly less than

$$(p'_{x+n}-p'_x).(1+\mathrm{A}'_{x+n}).$$

But this will cease to be always true when ϕ_x becomes less than $d'-d$, as also when ϕ_x is a quantity increasing as the premium increases.

Let us now briefly compare the Northampton table at 3 per cent., and the Carlisle table at $3\frac{1}{2}$ per cent., taking the former as a specimen of a factitious table, and the latter as a sample of a true one. In this case, then, $d'-d=$ ·00469, and the value of ϕ_x at given ages is as follows:—

Age.		Age.	
25	00799	50	01041
30	00819	55	00908
35	00869	60	00709
40	00916	65	00890
45	01003	70	00758

being always greater than $d'-d$, whence (ϕ_x+d-d') is positive, and $1+\mathrm{A}'_x$ must always exceed $1+\mathrm{A}_x$. But it will be seen that ϕ_x sometimes increases with the age, and sometimes decreases; so that $(p'_{x+n}+\phi_{x+n})-(p'_x+\phi_x)$ will be sometimes greater and sometimes less than $(p'_{x+n}-p'_x)$. The effect of this is to make the liability, as determined by the factitious table, in some cases too little and in others too great. But, by a parity of reasoning, the additions made by way of bonus will always be too little, and, what is worse, will be so in a greater degree at some ages than at others, while at subsequent valuations these very additions will again be over-estimated, but in different degrees.

If in any particular case it be found that the average rate of interest annually realised has been 3 per cent., then the use of the Northampton table in making valuations must lead to still worse consequences. For d being then equal to d', the relation between the true and factitious annuities will be expressed by

$$1+\mathrm{A}_x=\frac{1+\mathrm{A}'_x}{1+\phi_x(1+\mathrm{A}'_x)}.$$

Whence the latter will always be less than the former so long as ϕ_x has any positive value. Let the true table be that called the "Experience;" or, what amounts to the same thing, let the value of ϕ_x, at the undermentioned ages, be

Age.		Age.	
25	00738	50	00695
30	00762	55	00551
35	00781	60	00341
40	00793	65	00090
45	00761	69	00006

Then it will be seen that whilst $(p'_{x+n}-p'_x)$ is nearly the same as $(p'_{x+n}+\phi_{x+n})-(p'_x+\phi_x)$, $1+\mathrm{A}_{x+n}$ must be less than $1+\mathrm{A}'_{x+n}$; and that when these last approach to equality, $(p'_{x+n}+\phi_{x+n})-(p'_x+\phi_x)$ is much less than $(p'_{x+n}-p'_x)$. Thus it appears that, except in extreme cases, the values of assurances will at every age, and after any given time, be considerably greater by the true table than by the factitious one; and hence the total amount reserved for liabilities, as determined by the Northampton table,

will, under such circumstances be very much less than it ought to be.* These inconsistencies, it must be remembered, are not merely those between two tables selected at hazard, but result, in fact, from the introduction of a quantity into the investigation which cannot, with any degree of propriety, be mixed up with it. In other words, they arise from making valuations in accordance with the data derived by a direct process from the premiums charged; a practice which, however excusable in the infancy of life assurance, has nothing to justify a perseverance in it at the present day.

All this, however, serves only to confirm what has been said by the best authorities when speaking of these erroneously-constructed tables. Thus Mr. Gompertz, in a paper read before the Royal Society, in 1820, and quoted by Mr. Griffith Davies in his able work on life assurance, observes:—"That the proper method of regulating the premiums for life assurances, &c., is to employ what may appear to be the correct average rates of interest and mortality *to form the calculation upon,* and to the results, to make such additions as may leave an adequate portion for the security, profit, and expenses of the Insurance Company;" adding, "That it does not seem possible to adopt tables of mortality which are not correct in themselves, connected with a rate of interest which is not the average rate made in reality, so that the advantage may tend in one direction." Mr. Sang also, in one of his essays read before the Edinburgh Society of Arts, says, with his usual vigour of thought and expression:—"If, in this business, we use a life table

* This observation will apply with double force to instances in which no reserve is made beyond the amount so determined. For it must be carefully noted that the comparison here is with the liability indicated by a true table, and that it is invariably the practice to make a large addition to the amount of liability indicated in that way. Thus, after 20 years, the values of assurances by the Northampton 3 per cent. and Experience 4 per cent. tables are as follows:—

Age at Commencement.	Northampton 3 per cent.	Experience 4 per cent.
24	21·4	20·2
29	24·4	24·4
34	27·6	29·1
39	31·2	34·2
44	35·6	39·5
49	41·2	44 7
54	47·7	49·7
59	54·1	54·3
	283·2	296·1

The latter being, in the aggregate, a good deal in excess of the former, although the rate of interest is higher. But the advocates of this method would, nevertheless, consider a large additional reserve necessary, whereas those of the former would, as a general rule, contemplate no other than that here exhibited!

known to be erroneous, we can neither tell the amount of the error nor the side on which it is likely to be, except by contrasting the results with those drawn from a life table believed to be nearer the truth; and we are thus placed in the awkward position of virtually making two valuations and using the bad one, augmenting the absurdity, perhaps, by correcting the erroneous results, so as to bring them to the true ones."

Lastly, Mr. Babbage, speaking to the same point, says:—"This method of determining the premiums to be taken is, in the present state of our knowledge, both inexpedient and unscientific, although when first employed it deserved a very different character. I imagine it will not require much argument to show that the more proper course would be to determine, as nearly as we can, the real value of the risk, and consequently the amount of premium just sufficient to meet it, and to add to this such percentage as will defray the expenses of management, &c."*

When the table of premiums so well known as the Northampton table was first constructed, it is probable that its framers did not anticipate that the payments to be made in accordance with it would be more than sufficient to provide for the sum assured. The addition of 15 per cent. originally made to the premiums, would rather lead to the inference that they were considered insufficient without it to meet other contingencies. At the outset, then, we find established the relation between the sum guaranteed under contract and the premium derived from what were then supposed to be the true average rates of mortality and interest: the 15 per cent. being, no doubt, disregarded in any official calculations, or at least dealt with separately, since we have never heard of any table of annuities which would, by a direct process, give Northampton premiums plus 15 per cent. When the payments

* Since writing this paper, my attention has been called by Mr. Peter Gray to the following observations made by Mr. Babbage in his evidence on Friendly Societies (*Report*, 1827, pp. 29, 30):—"If I am asked which tables I should wish to use in making any calculations relative to the poorer orders of society, I should state that my object would be to get such tables as exactly and perfectly represented those classes throughout all their ages; but it may be said that this is unsafe, and that others should be taken where the deaths are more numerous than those that really happen. My view in all cases is, let us get, as nearly as we can, the law of mortality of the class for which we want to calculate, and add to the prices computed from it some proportional part sufficient to insure the safety of the establishment which uses them. I strongly object to using tables giving a greater mortality than is expected to take place, a course which has sometimes been defended on the ground of safety to the establishment. Safety is much more certainly secured by judging, as nearly as possible, the true risk, and adding an additional sum for security. If tables not representing the mortality of the class for whom they are designed are employed, every step in the reasonings that are deduced from them is liable to increased error; and if the calculations are at all complicated, the errors so introduced may not improbably act on the opposite side to that which they were introduced to favour."

thus adjusted were found, as they soon were, to be more than sufficient, it is evident that the consistent course was to revise the rates of mortality and interest. The circumstance of a surplus being found, to arise uniformly throughout a given period, proved incontestably that the premiums charged for the risk were excessive, and consequently that some error existed in the assumed rate of mortality or in that of interest. Not to be guided by such evidence was to persevere in error, and to neglect the most direct indications of being in it. It would behove those concerned, then, to determine in what the error consisted, or wherein a false assumption had been made, and to correct it accordingly, so that thenceforward no surplus should arise in respect of the premium for the risk. It by no means followed, however, that the excessive portion of the premium should cease to be charged. It could of course be retained in its new character as a mere guaranty against adverse fluctuations, to be relinquished, if so arranged, at stated intervals, with such deductions as might be necessary, or with such additions as circumstances might warrant. But it will be perceived that it no longer formed any part of the true premium, and was therefore quite out of the pale of investigations dealing solely with the risk and with the payment required to meet it. Hence the premium actually charged might thenceforth be regarded as consisting of two distinct parts, the one being simply a provision for the sum guaranteed under the policy, derived by a direct process from the best authenticated rates of interest and mortality; the other, a merely arbitrary and independent addition, providing against contingencies of a totally different character, and therefore the subject of a treatment altogether distinct. In this way p_x would come to be looked upon as $p'_x+\phi_x$; and the elements which, at the outset, corresponded with p_x, would finally adjust themselves, so as to yield p'_x, ϕ_x being thus excluded, as it were, from the arena within which the theorems of the doctrine of probabilities were to have their legitimate play.

Thus, then, we again arrive at the conclusion that whatever the rates of premium charged may be, *the difference between them and the true premium must be discarded, or dealt with apart in all official calculations, if we would have anything like consistency in our deductions.* What the amount of the true premium may be under any given circumstances, can only be got at by close and careful investigation; but at the present day we can have no great difficulty in arriving at an approximation, which will at least show that some distinction is to be drawn between the two, and which will

extricate us from the inconsistencies invariably arising whenever we persist in confounding the one with the other.

From what has been said we shall be enabled, I think, without leaving room for much difference of opinion, to arrive at the true principles which should govern us in the distribution or division of surplus; for, since the payments of the true premium will evidently, from their nature and character, be always absorbed in the amount which constitutes the liability, whatever surplus is found at any time to exist must clearly arise from payments of ϕ_x, and the interest upon them; and therefore nothing more remains to be done than to redistribute these last to the several contributors. In short, we may imagine that when the payments of p_x are made, p'_x is carried to one fund and ϕ_x to another; and that interest being properly apportioned between the two, the remainder of the latter fund, after payment of expenses and other charges, is returned, at given intervals, in due proportion to those who have created it; the fund arising from the payments of p'_x, and the interest upon them, being of course sufficient to meet the claims as they fall in: so that if we call S the real surplus found to exist after any valuation, and $\Sigma.\phi_x a^n$ the artificial one determined from the payments of ϕ_x improved at the true rate of interest, then the share of each contributor may be represented by $\frac{S}{\Sigma.\phi_x a^n} \times \phi_x a^n$; this latter quantity denoting the amount of the surplus payments actually contributed by each.

This method will have the effect of distributing the surplus with probably as much fairness as we can expect practically to arrive at. But that it is but a near approximation will be observed, when it is considered that, for the expenses to be duly apportioned, the collateral payments of ϕ_x must each extend through the same period, a condition which will rarely obtain. The methods usually adopted, however, at the present day are objectionable, as involving other inaccuracies beside the one mentioned. I will select two which are most in vogue, viz., the division in proportion to the amount, at compound interest, of the premiums paid, and that in proportion to the same amount after deducting the value of the assurance. Now, it will be remembered that the total liability corresponds with the amount at the true rate of interest of the payments of the true premium, less the claims; whilst the surplus owes its existence to the accumulations of ϕ_x, less the expenses, which, for our present purpose, we may express as follows, viz.:—

$$\Sigma.p'_x a^n - \Sigma.c = \Sigma.l = \text{the liability},$$
$$\Sigma.\phi_x a^n - \Sigma.e = \Sigma.s = \text{the surplus}.$$

The division, then, in accordance with the one principle, will be in the proportion which $\Sigma.p_x a^n$ bears to $p_x a^n$; and that in accordance with the other in the ratio expressed by $\Sigma.p_x a^n - \Sigma.l : p_x a^n - l$; that is to say, they will be in the ratio respectively of

$$\Sigma.p'_x a^n + \Sigma.\phi_x a^n : p'_x a^n + \phi_x a^n, \text{ and}$$
$$\Sigma.\phi_x a^n + \Sigma.c \quad : \phi_x a^n + e, \text{ or } p_x a^n - l.$$

But it has been shown that the surplus arises from $\Sigma.\phi_x a^n$ alone, and not from either of the sources above denoted, and hence it will be seen that the ratios adopted in these two methods are, under ordinary circumstances, neither of them properly applicable.

I do not propose, on this occasion, to consider the several modes in which the share of surplus may be returned or placed to the account of the assured; but before I conclude I would say a few words as to the nature of the addition made to the true premium—that is, as to the relation which ϕ_x should bear to p'_x. In the Northampton table we have seen that the former is an irregular quantity, if anything, tending to decrease as the age increases. If the true mortality be represented by the Carlisle table, and the true rate of interest by 4 per cent., then ϕ_x in the Northampton table is nearly constant. But the usual practice is to make the addition a percentage on the premium, the effect of which is to make the entrants, at all ages, contribute to extra contingencies in proportion *to the intensity of the risk,* without any reference to the time relatively occupied in completing the transaction, or to the natural duration of the assurance.* But in the light in which we have been regarding these extra payments, there is no one object to which they are devoted, or purpose for which they are required, which is not of periodical recurrence; and therefore it would seem that the equity of the case would be better met by making the assurers contribute to extra contingencies in proportion to the time and the sum, and not in the proportion above stated; that is to say, ϕ_x should be a constant quantity,† varying only with the

* This is obviously unjust, as well as unmeaning! Whatever is paid over and above the premium for the risk is in the nature of a contribution to rent, taxes, salaries, &c. &c.; and what more simple or equitable plan can be devised than the requiring the like annual contribution to these charges on account of each one hundred pound assurance? A member's policy does not entail greater expense on the Society because he happens to be older than another!

† See Professor De Morgan's *Essay on Probabilities*, page 271.

sum assured, and of such an amount as the objects of the assurers may induce them to make it.*

It will, perhaps, be necessary, however, to admit of a certain departure from this principle where commission is usually allowed to agents. This charge, it is well known, instead of being regulated, as it ought evidently to be, by the amount assured only, is always in proportion to the premium, and therefore it will be proper to take this circumstance so far into consideration as to make a separate addition to the premium on account of it. The premium charged will thus consist of the quantities $p'_x+\phi+\frac{p'_x+\phi}{19}$ or $\frac{20}{19}\left(p'_x+\phi\right)$, the rate of commission allowed being 5 per cent.; and it will be found that, in a table so formed, the abnormal and distorted features so frequently presenting themselves in the rates for assurance put forward from time to time are no longer to be met with.

On Weights and Measures. By Professor Leoni Levi.†

PUBLIC attention has for some time past been earnestly directed to the introduction of the decimal system in our weights, measures, and coins. The nation is generally convinced, that the adoption of such a system would prove of immense benefit—that it would afford great facilities for calculations of all kinds, that it would shorten the work of education, that it would economise labour, and that it would diminish the chances of error. The Society of Arts, and other scientific societies, have investigated the subject in all its phases and bearings, and we have been expecting the speedy adoption of some practical plan which would be certain to confer so great a boon. Unfortunately, the Russian war, the Indian mutiny, and other political events, have rendered it necessary to put aside the consideration of many social reforms, and this, among the rest, shared the same fate. We have bestowed, also, far too much attention to the pound and mil scheme, as if upon it rested the entire

* If S denote the sum resulting from the addition of the true premiums *per cent.*, taken at the ages 21, 22, 23, 24, &c., to 65, and if it be proposed to increase them one third, then $\phi=\frac{1}{45}\times\frac{S}{3}$, or $\frac{S}{135}$; and the total annual contribution to extra contingencies, irrespective of commission, will always be equal to $\frac{\phi}{100}$. (total sum assured).

† Reprinted, by permission, from No. 9 of *The Exchange*.

question of decimalisation, and thus years have passed without a single step of a definite character being taken.

From this state of torpor, however, we are suddenly awakened, by the publication of a large Blue-book, being the Report of Mr. Ewart's Committee of the House of Commons "On the practicability of adopting a simple and uniform system of weights and measures, with a view not only to the benefit of our internal trade, but to facilitate our trade and intercourse with foreign countries." There is one advantage in the present Report, that it leads us to the bottom of the question, and enables us to form a correct view of the entire system. Hitherto we have only heard of decimal coinage, and we have omitted to realise the close relation of the coinage to weights and measures. It was well to direct our attention to these. Indeed, we quite agree with Lord Overstone, that we should take a comprehensive view of the whole before we begin to make any change:—

> "A decimal coinage," said Lord Overstone, in his Draft Report for the Decimal Coinage Commissioners, "is one element of a complete decimal system, in the same sense in which the florin is one element in a decimal coinage; and to decide in favour of a change in the coinage without determining what would be the best course to take with reference to other parts of the metric system, would be to fall into the same mistake in a far more serious and important shape than was committed in issuing florins in their present form before any final adjudication on the general question of decimal coins. Any precipitate decision on an isolated part of what is really one great question is the more earnestly to be deprecated, because a change once introduced into the coinage will be irrevocable, and ought not to be regarded until every consideration bearing upon the question has been fully entered into. When we consider the intimate relation between weights and measures and coins—that is, between the instruments by means of which commodities are divided and distributed, and the instruments by which the commodities so divided are paid for—it seems scarcely credible that a proposition can be seriously made to introduce, at the cost of much temporary inconvenience, a new system of coinage, without reference to the nature or principle of that simplification of our weights and measures which the anomalies of the present system appear to render indispensable."

Nor are those anomalies light and unimportant. The bare description of them will suffice to establish an absolute necessity for a prompt legislative remedy. We can afford to wait for the introduction of a decimal coinage, but we cannot delay to introduce uniformity in our weights and measures. As many as ten different systems are actually in use in this country. There is, first, the grain, computed decimally, which is used for scientific

purposes; second, the troy weight, under 5 Geo. IV., c. 74, and 18 and 19 Vict., c. 72; third, the troy ounce, with decimal multiples and divisions, called bullion weight, under 16 and 17 Vict., c. 29; fourth, bankers' weight, to weigh 10, 20, 30, 50, 100, and 200 sovereigns; fifth, apothecary weight; sixth, diamond weight and pearl weight, including carats; seventh, avoirdupois weight, under 5 Geo. IV., c. 75, and 18 and 19 Vict., c. 72; eighth, weights for hay and straw; ninth, wool weight, using as factors 2, 3, 7, 13, and their multiples; and tenth, coal weight, decimal, under 1 and 2 Wm. IV., c. 76, and 8 and 9 Vict., c. 101. We have also in occasional use the weights of the metric system. For measures of length we have the ordinary inch, foot, and yard. In cloth measure, we have yards, nails, and ells. There are four different sorts of ells. For nautical purposes we have fathoms, knots, leagues, and geographical miles, differing from the common mile. The fathom of a man of war is 6 feet; of a merchant vessel, 5½ feet; of a fishing smack, 5 feet. We have also the Scotch and Irish mile, and the Scotch and Irish acre. There are several sorts of acres in the United Kingdom, and there are a great variety of roods. We have in almost every trade measures of length specially used in those trades. For the measurement of horses, we have the hand; shoemakers use sizes; and we are compelled to adopt gauges where the French use the millimetre. The gauges are entirely arbitrary. The custom of the trade is the only thing which would decide the question in case of dispute. For measures of capacity, we have twenty different bushels. We can scarcely tell what the hogshead means: for ale, it is 54 gallons; for wine, 63. Pipes of wine vary in many ways; each sort of wine seems to claim the privilege of a different sort of pipe. For measures of weight, we have about ten different stones; a stone of wool at Darlington is 18 lbs., a stone of flax at Downpatrick is 24 lbs., a stone of flax at Belfast is 16¾ lbs., but it is also at Belfast 24½ lbs.; having in one place two values. The cwt. may mean 100 lbs., 112 lbs., or 120 lbs. If you buy an ounce or pound of anything, you must inquire if it belong to Dutch, troy, or avoirdupois weight. Such are only a few of the many anomalies exhibited by our weights and measures. Can anyone say that such a state as this is creditable to our civilisation, or compatible with our extensive commercial interests? It cannot be said that we are deficient in laws on the subject—indeed, as far as legislation is concerned, there has been a superabundance of it. And yet we are as distant as ever from this great desideratum. What, then, is the reason?

What is the cause that we are behind almost every other nation in this most essential object? Partly because the law has left many loopholes whereby local usages have been allowed to remain; but much more because, in the want of a clear and intelligible system capable of meeting the manifold requirements of society, every trade and every class had to seek its own way of weighing and measuring commodities.

In dealing with a question of such a momentous and practical character, the Committee of the House of Commons had three alternatives—first, to retain the present system, though defective, and simply to endeavour to amend the law—as it has been done again and again, without result—a mode which no one recommended; second, to create a separate decimal system of our own—that is to say, to decimatise our present units, the pound, the yard, and the gallon—as recommended by Professors Airy, De Morgan, and others; third, to adopt, in common with other countries, the metric system, in accordance with the recommendations of a host of witnesses. The first of these modes appeared to the Committee quite useless. The second would necessitate a complete change, and cause much confusion and trouble, without corresponding results. The latter—the introduction of the metric system—seemed by far the simplest and best. By adopting this course, the Committee were enabled to set aside the chaos of our irregular method, and all the discrepancies by which it is distinguished, and to enter at once into a plain and symmetric principle. And when they considered that their instructions were not only to seek a system perfect and complete itself, but one which should facilitate trade and intercourse with foreign countries, they could not well hesitate in their choice.

The metric system has the great advantage of being in existence and in great favour in many countries. France, Holland, Belgium, Italy, Spain, Portugal, Switzerland, and Greece, and several countries of South America, all have adopted the same. The public mind has long ago pointed to this system as the most convenient for all purposes, and the only basis of international uniformity—a matter of great importance in these days of international movements. From all sides we hear expressions of a want of such uniformity. Our trade with the continent has immensely increased since the conclusion of our treaties of commerce. For some time past our commerce with the countries using the metric system has increased much more rapidly than our commerce with countries using the English or other systems of weights and measures. Our

exports to countries using the metric system, in 1847, amounted to £23,696,000; in 1861 it had reached £55,242,000—showing an increase of 133 per cent.; and our exports to countries using the English system, in 1847, amounted to £16,261,568; and in 1861 to £24,211,449—though it would have been a larger amount had it not been for the American war, which has reduced our exports to that country more than £10,000,000. The Associated Chambers of Commerce of the United Kingdom, at their annual meeting in 1861, speaking as delegates on behalf of their various districts, and representing some of the most important towns in the country, as well as various branches of industry, unanimously passed the following resolution:—"It is highly desirable to adopt the metric system, which has been introduced into many European countries with great advantage to the saving of time in trading and other accounts." The Jurors of the International Exhibition found difficulty in arriving at a common standard, in consequence of the various weights and measures used by the exhibitors of different countries. The International Statistical Congress has again and again expressed an opinion in favour of international units of weights and measures; and our men of science complain of the confusion arising in scientific observations from the uncertainty of the weights and measures used by observers in different countries. "The majority of the facts in natural history," said Professor Owen, in his speech as a member of the deputation to the Board of Trade, "include the elements of weights and measures. An English anatomist and physiologist gives the weight of the brain, lungs, &c., in relation to the weight of the body of some rare animal which he has had the opportunity to examine: the foreign physiologist desires to reduce the English weights to those of his country. If the kind of weights used by the Englishman be not specified—viz., avoirdupois or troy—the description is useless to the foreigner in regard to the important constants of the proportion of parts or organs to the whole body. Similar difficulties are experienced as regards linear measures. And although, when the system of weights or measures is noted by the observer, its reduction, or the finding the equivalent in another system, may be a small demand upon his time, yet the repetition of that act takes a serious amount from the working hours of the individual; and when multiplied by the number of observers, obstructed by conflicting systems of weights and measures, the impediment to the progress of the sciences of observation becomes so great as to render the subject quite worthy of the consideration of legislative

authority." These and other facts of a like nature were present to the mind of the Committee of the House of Commons, and were enforced by the evidence of some of the most distinguished men from various countries. Seldom, indeed, have we seen such a galaxy of eminent witnesses as appeared before Mr. Ewart's Committee—men such as Mr. Graham, the Master of the Mint; Mr. Fairbairn, the late President of the British Association; Professor Airy, the Astronomer Royal; Professor Miller, of Cambridge; Professor De Morgan, Dr. W. Farr, Mr. James Yates, Fellow of the Royal Society; M. Michel Chevalier, Senator, Member of the Institute; M. Visschers, Conseiller des Mines, of Belgium; Dr. Karmarsh, Principal of the Polytechnic Institution, of Hanover; Dr. Steinbeis, President of the Board of Trade, of Wurtemberg; and many others. And seldom, too, have we found such a concurrence of opinion as was exhibited by them. We do not wonder that they produced like unanimity in the resolutions of the Committee.

Let us now examine the nature of the metric system, and ascertain what relation it bears to our own. It had long been the desire of France to have a uniform system of weights and measures. But under the old regime there was too much indolence, and too great a disregard of material interests, to allow them to think of such things. As soon, however, as the nation awoke to a sense of its own importance, the *Assemblée Constituante* took the matter up in earnest, and on May 8, 1790, it passed a resolution desiring the King to obtain the co-operation of the English Legislature with the National Assembly for the determination of a natural unit for the comparison of weights and measures. The proposal was, that an equal number of commissioners, chosen from the Academy of Sciences and from the Royal Society, should meet, in order to find, at the parallel of latitude half way between the equator and the pole, or any suitable parallel, the length of the second's pendulum. No response having been given by Britain to this invitation, the Academy proceeded in their labours by themselves, and their first decision was that all the multiples and subdivisions of the system should be according to the decimal scale, and that the units of surface, capacity and weight, should all depend on the unit of length. The Academy then appointed as commissioners, Lagrange, La Place, Borda, Monge, and Condorcet. These commissioners, having carefully discussed the relative merits of the invariable length which is known to be required for the exactness of a second's pendulum at any given latitude, and of a unit taken from the

dimensions of our planet, decided upon preferring the latter, as not involving the heterogeneous element of time, and being also necessarily of a more cosmopolitan character. The ten-millionth part of the arc of the meridian comprised between the equator and the north pole was therefore selected as the unit of linear measure. At first the commissioners considered it quite sufficient to take the standard of length from some previously executed measurement of the earth; but eventually they resolved to undertake all the preliminary work afresh, and the astronomers Delambre and Mechain were appointed to undertake the geodesical operations. We shall not attempt to follow these astronomers in their arduous labours, or to detail the strange adventures which they experienced during the worst times of the French revolution. It is sufficient to say that ten years elapsed before the measurements terminated; but as soon as they were concluded, the French Government again issued invitations to neutral and allied countries to send deputies to assist in the final settlement of the metric system, so that it might be adapted to the usage of all nations. It was then that the exact measure of the metre was fixed, and the other standards of weight and capacity also definitively settled. For the standard of surface for land measure, the commissioners took the square of ten metres on each side, or one hundred square metres. The standard of capacity for liquids was determined by finding a cylindrical volume equal to a cube, whose edges are formed by tenths of the linear standard. This is the litre; and the weight of a litre of distilled water at its maximum density was adopted for standard of weight, and called a kilogram.

Having once adopted the units, the commissioners proceeded to determine the multiples and subdivisions, according to the decimal scale, and they also resolved upon a nomenclature totally new and distinct from any in use—taking Latin and Greek words for ten, hundred, thousand, and ten thousand, to express respectively the decimal gradation, upward and downward, of the respective units. It was, perhaps, unfortunate that the commissioners adopted such a nomenclature. We are quite sure that it retarded the popular adoption of the system. But the commissioners objected to take words derived from the vernacular tongue—not on account of any preference for the Latin or Greek, but because they expected they would excite less prejudice, and would be more readily adopted by other nations. There is, however, great value in the fact that each name expresses exactly and clearly, not only the quantity therein represented, but the relation in which it

stands to all the other quantities of the same. Indeed, a glance at the system shows at once how superior it is to our own uncouth and irregular gradation, with binary and duodecimal factors, and a considerable admixture of other modes of progression. The units of measures and weights are as follows :—The *Metre,* the unit of the measure of length, is the ten-millionth part of the fourth of the terrestial meridian : the *Are* is the unit of superficial measure, and is a square of ten metres on each side ; the *Litre,* the unit of capacity, is a cubic decimetre ; the *Stere,** the unit of the measure of solidity, is a cubic metre ; and the *Gram,* the unit of weight, is the weight of a cubic centimetre of distilled water at the temperature of its greatest density. For multipliers, there are added to the respective units the following Greek prefixes :—*Deca,* viz. 10 times ; *Hecto,* viz. 100 times ; *Kilo,* viz. 1,000 times ; *Myria,* viz. 10,000 times. For divisors, there are added to the respective units the following Latin prefixes :—*Deci,* viz. 10th part ; *Centi,* viz. 100th part ; *Milli,* viz. 1,000th part. But perhaps a tabular statement will bring the whole out even more clearly :—

NOMENCLATURE OF WEIGHTS AND MEASURES AND THEIR EQUIVALENTS.

Measure of Length.

Names.	Value.	English equivalents.
Millimetre	The thousandth part of a metre	0·03937
Centimetre	,, hundredth ,, ,,	0·39371
Decimetre	,, tenth ,, ,,	3·93708
METRE	——	39·37079
Decametre	Ten metres	32·80916 feet
Hectometre	Hundred do.	328·09167 ,,
Kilometre	Thousand do.	1093·63890 yards
Myriametre	Ten thousand do.	10936·38900 ,, or 6 miles 1 furlong 28 poles

Land Measure.

Centiare	The hundredth part of the are, or square metre	1·1960 square yards
ARE	——	119·6046 ,,
Decare	Ten ares	1196·0460 ,,
Hectare	Hundred do.	11960·4604 ,, or 2 acres 1 rood 35 perches

Measure of Capacity.

Millilitre	The thousandth part of a litre	0·06103 cubic inches
Centilitre	,, hundredth ,, ,,	0·61028 ,,
Decilitre	,, tenth ,, ,,	6·10280 ,,

* The Stere, used in France for measuring stacks of firewood, is not wanted in England.

Measure of Capacity (continued).

LITRE	——	61·02803 cubic inches, or 2·1135 wine pints
Decalitre	Ten litres	610·28028 cubic inches, or 2·642 wine gallons
Hectolitre	Hundred do.	3·5317 cubic feet, or 26·419 wine gallons
Kilolitre	Thousand do.	35·3171 cubic feet, or 1 tun 12 wine gallons
Myrialitre	Ten thousand do.	353·17146 cubic feet

Measure of Solidity.

Decistere	The tenth of a stere	3·5317 cubic feet
STERE	——	35·3174 ,,
Decastere	Ten steres	353·1741 ,,

Weights.

Miligram	The thousandth	0·0154 grains
Centigram	,, hundredth	0.1543 ,,
Decigram	,, tenth	1·5434 ,,
GRAM	——	15·4340 ,,
Decagram	Ten grams	154·3402 ,,
Hectogram	Hundred do.	3·527 oz. avdpois.
Kilogram	Thousand do.	2 lbs. 3 oz. 4½ drs. avdps.
Myriagram	Ten thousand	22·0485 lbs. ,,
Quintal	. . . . .	1 cwt. 3 qrs. 25 lbs.
Ton	Thousand kilograms.	

This is the system recommended by the Committee, and we congratulate the nation on the fair prospect of a satisfactory and permanent settlement of this great question. Any further patching up of the present method would, we are sure, have proved fruitless.

The recommendations of the Committee are as follows :—

"1. That the use of the metric system be rendered legal, though no compulsory measures should be resorted to until they are sanctioned by the general conviction of the public.

"2. That a Department of Weights and Measures be established in connection with the Board of Trade. It would thus become subordinate to the Government, and responsible to Parliament. To it should be intrusted the conservation and verification of the standard, the superintendence of inspectors, and the general duties incident to such a department. It should also take such measures as may from time to time promote the use and extend the knowledge of the metric system in the departments of Government and among the people.

"3. The Government should sanction the use of the metric system, together with our present one, in the levying of the customs duties; thus familiarising it among our merchants and manufacturers, and giving facilities to foreign traders in their dealings with this country. Its use, com-

bined with that of our own system, in Government contracts has also been suggested.

"4. The metric system should form one of the subjects of examination in the competitive examinations of the civil service.

"5. The gram should be used as a weight for foreign letters and books at the post office.

"6. The Committee of Council on Education should require the metric system to be taught (as might easily be done, by means of tables and diagrams) in all schools receiving grants of public money.

"7. In the public statistics of the country, quantities should be expressed in terms of the metric system in juxtaposition with those of our own, as suggested by the International Statistical Congress.

"8. In private Bills before Parliament, the use of the metric system should be allowed.

"9. The only weights and measures in use should be the metric and imperial, until the metric has generally been adopted.

"10. The proviso in the 5th and 6th William IV., c. 63, s. 6, should be repealed.

"11. The department which it is proposed to appoint should make an annual report to Parliament."

According to these recommendations, the first thing to be done is the legalisation of the metric system. It is needless to say that, under the present law, a sale of wheat by the kilogram, or of cloth by the metre, is illegal, and could not be enforced. It is now proposed to make all contracts embodying the weights and measures of the metric system legal. Yet it is only the permissive use which is to be authorised, and no one is to be compelled to use the metre instead of the yard, or the kilogram instead of the pound. The creation of the Department of Weights and Measures is absolutely necessary. At present, the Comptroller of the Exchequer has some superintendence over the weights and measures, but it is of a very slender kind. The Inspectors of Weights and Measures, appointed by the magistrates of different towns, have nothing to do with the central office. They only report to the magistrates, and with a divided responsibility there is no guarantee whatever for the due enforcement of the law. If the metric system is to be adopted, there is great need, moreover, of diffusing information on the subject, in the shape of comparative tables and scales. The new standards must be introduced and supervised. Measures must be taken for the verification of the standards, and for the allocation of local standards. All this requires care, and can only be done by officers appointed for the purpose. The next recommendation of the Committee is to the effect that customs duties may be levied according to the metric system. Continental traders must find it very difficult to calculate the amount of English duties

on their goods. Let us remove this difficulty by issuing our tariff, not only in the present form, but reduced according to the metric system. This will facilitate trade, and greatly aid in familiarising merchants with the new weights and measures. A recommendation of a practical nature is also made to allow the use of the gram, in the post-office, for foreign letters. Sir Rowland Hill gave many illustrations of the complicated and inconvenient operation of the different weights in different countries in the postage of letters. In Germany, for instance, the unit of weight is the loth, equivalent to $257\frac{5}{10}$ths grains; in this country it is the half-ounce, equivalent to $218\frac{75}{100}$ths grains—showing a difference of 17 per cent. And the consequence is, that though the same rates are charged in both countries, the German correspondent has an advantage, and it becomes cheaper to send a letter unpaid than prepaid. With France, again, there is another difficulty, arising from the difference in the scale of internal postage. France proposed to adopt a 10-gram scale, in lieu of their $7\frac{1}{2}$-gram, as the unit of weight for letters to England, with a view to uniformity; but we have no weight equivalent to 10 grams, or a third of an ounce—and the 15 grams, equivalent to our half-ounce, is too heavy. The adoption of the gram will smooth this difficulty. The Committee suggested other special means (educational and otherwise) for diffusing information of the subject, and recommended also the abolition of a clause in the present law, which shelters many of the present irregularities in our measures. It is pleasing to see, in the whole of this Report, a perfect combination of what is purely scientific with all that is eminently practical.

The Committee have fully appreciated the difficulties connected with any changes in weights and measures. If they have come to the determination to recommend the adoption of the metric system, it is because, all things considered, it is the best that could be offered. But in doing so, they were alive to the necessity of great caution in adopting such a new system, and, taking the example suggested by the experience of other countries, they have deemed it necessary to stimulate the voluntary adoption of the system, before any compulsory measure is resorted to for the enforcement of it. But why, after all, should we be so apprehensive of changes? Changes have been made before this, and they have been attended with little or no inconvenience. What have other nations done when they introduced their new systems? What have the United States done when they abandoned their pound for the dollar? What has Ireland done when the Irish currency was abolished? They

faced the difficulties, and conquered them. We are not underrating the inconvenience of the change. The weights, measures, and coins of the realm are so identified with national ideas, and embodied in our mental vocabulary, that no change can be made in them without producing a sudden disturbance. But with the means now in existence for diffusing information, any ignorance of the subject will soon be dispelled. Newspapers, magazines, and reviews will, by means of letters, articles, and dissertations, throw a flood of light on the question in all its bearings. The teachers in our British and National schools will enter in right earnest into the tuition of the decimal system. The merchant will not be slow in reducing the prices to the new denominations, and hard necessity will soon teach the humblest individual new ideas of quantities and of value. There is no reason, therefore, for fear and hesitation. If we can answer affirmatively, that the change is wanted, and that the reform is beneficial, we may courageously give ourselves to carry it out with as little delay as possible.

NOTES AND QUERIES.

Suggestion by Mr. Newmarch as to the Federation of certain cognate Societies.—I have long entertained the opinion that the time has come when it will be found advantageous, and perhaps necessary, that the six or seven societies now existing in London for the cultivation of different branches of social science, should form themselves into a Federation, not so complete as to be subversive of individual independence, but sufficiently compact to secure the great objects of (1) concentrated libraries and places of meeting; (2) economy in management and expenses; (3) moral and intellectual power arising from the combination of several parts into one consistent whole.

Several members of the council will remember that at various times during the last two years, I have suggested the desirableness of an arrangement of the kind now indicated, and that some progress has been made in the consideration of details.

I employ, without any hesitation, the phrase *Social Science*, not perhaps as the most exact term that could be found, but as the title of a new branch of knowledge which has already acquired, in the public apprehension, a definite place and a recognised function.

There are in London at the present time the following seven Societies, all engaged, in one way or the other, in the cultivation of social science, viz.:—

1. Statistical Society.
2. Institute of Actuaries.
3. Juridical Society.
4. Society for Amendment of Law.
5. Reformatory Union.
6. Association of Sanitary Officers.
7. National Association for promoting Social Science.

It seems to me that the manifest policy of these seven separate Societies—to say nothing of manifest duty—is to form themselves into a powerful "Institute of Social Science," on the model of the British Association and the Social Science Association—that is to say, full sectional action and independence under the supervision of a central authority.

It will be observed that, in the list of Societies just given, there is no provision for the investigation and discussion of questions of *Economic Science* as a separate and special pursuit; and yet sound economic views are indispensable to the successful treatment of most of the subjects which engage the attention of the learned bodies now enumerated. There is, moreover, the striking anomaly, that in the native land of political economy, and in the country which has done, and is doing, the most to discover its laws and illustrate their application, there is in the multitude of scientific associations not one which specially cultivates a branch of knowledge so essentially English and practical. In France there have been for a long period the Academy of Moral and Political Sciences, besides other special means of promoting economical studies. An Institute of Social Science would be well able, by means of concentrated strength and resources, to establish a separate section of political economy, and so supply a defect and a want which has been long confessed.

It may be sufficient to say here, that conformity to at least four principles may be assumed to be indispensable in any efforts which may be made to establish a federal union of Societies, viz., (1) that each existing Society shall remain in possession of its own property, shall continue to be governed by its own internal rules, and shall continue to choose its own managers and officers; (2) that similar independence shall be preserved as regards the control of the publication of its own papers and proceedings; (3) that each meeting of each of the federated Societies shall be open to the members of each of the other federated Societies, so as to concentrate upon each department the force of the entire body; and (4) that the authority to be exercised by the officers and council of the Federation itself should be limited to the purposes and objects rather of advising than of actively interfering with, the associated Societies.

It has been stated that the memorial to the late Prince Consort, to be erected at Kensington, will include a hall or college available for the use of learned Sócieties. If this statement should be verified, it is allowable to say that no plan would more happily fulfil some of the favourite schemes of the lamented Prince himself, than a union in his memory of those learned bodies which cultivate that social science which is so greatly beholden to him as a founder, guide, and expositor.

CORRESPONDENCE.

ON MR. SAMUEL YOUNGER'S PLAN FOR THE ASSURANCE OF INVALID LIVES.

To the Editor of the Assurance Magazine.

Sir,—I have been for some time expecting to see a notice in the *Assurance Magazine* of the remarkable plan for the assurance of invalid

lives, suggested by Mr. Morrice Black, of the London and Yorkshire Office, and I have read with interest the paper of my friend, Mr. Younger, in the Number just issued. It appears to me, however, that there is a fundamental error of principle which renders both of the proposed plans unsound, and which the palliative considerations set forth by Mr. Younger do not affect.

The payment for a contingency ought never to depend on the issue of the event. An ordinary assurance on a healthy life, it has often been pointed out, is a transaction in the nature of a wager—morally harmless, it is true, but still a wager upon the happening of an event unknown to either party. If the life survive the year of payment of the premium, the Company stands to win; if it die, the Company will lose: and as the latter event is antecedently less probable, the Company gives odds to the assured. But as soon as the year has expired the transaction is closed: the assured cannot say, "The life has survived, return me my money;" nor the Company, if the death has happened, "We will pay you back what we have received from you, and no more." It is true the Company has entered simultaneously upon many other similar transactions, and expects to be neither a loser nor a gainer on the whole extent of them; but that is its own risk, not the risk of the assured. The proportion of the premium to the sum assured—that is, the amount of the odds given by the Company—is settled according to what are erroneously termed the "laws" of mortality, which are really only the results of past experience as extensive as we can collect. Those "laws" promise no certainty of life equivalent to the odds given, but merely enable the Company to say at what rate they can afford to run the risk.

An assurance upon an invalid life differs in no respect from an ordinary assurance, except that the rate of odds the Company can afford to give is matter of more anxious calculation. Whatever that rate may be, when fixed upon an antecedent investigation into the case, it must not be disturbed when the contingency is at an end. If, upon an individual case, or any number of cases, the Company find the transaction result in a loss, they must abide it; if they have made profit, the profit is legitimate, having regard to the antecedently equivalent risk of loss. There cannot be an "error" or "mistake," for subsequent remedy, in the estimate put originally upon the risk.

It is a second fallacy, I conceive, to "assume that the fact of a person, whose life was considered an extra-hazardous one, living to the end of the 'expectation' period, is complete evidence that the medical opinion was wrong, and that no additional premium was necessary." Reflection will show that the "expectation of life" means nothing whatever. It is merely an arithmetical result. It does not import that any one person, or any number of persons, will live that term of years; but only that, if you add the years lifetime of a number of persons together, and divide by the number of lives observed, you obtain a given result. If some die n years earlier, others will die n years later. Those who live beyond have not really "exceeded their expectation;" if it were so, all would live till the last age in the tables, for the "expected" age at death increases with every year of life completed. The "expectation" is not an element in the premium to be charged, nor is it a measure of time available for any purpose whatever.

By way of illustration, take the case of a person aged 30, as set forth in

Mr. Younger's example, to whose age the medical adviser recommends an addition of 10 years to be made for the purpose of calculating the premium to be charged—thus removing him from a class whose supposed "expectation" is 34 years, to one whose supposed "expectation" is 28 years. Now, even if it were admitted that the payment should abide the issue of the event, that issue would not arise at the end of 28 years, or at the end of 34 years, as suggested by Mr. Younger, but at the end of such a period as would make the average duration of the assurance for all who have been exposed to the risk, including those who have died previously to the supposed expectation, not less than 28 years.

I see that Mr. Younger admits he has "departed from theory;" but I think that the experience of all Offices which have entertained invalid assurance would show that such departures on the wrong side are very dangerous. It is a most hazardous class of business, and the Company requires every practical and theoretical barrier against loss that can be devised. Every plan which charges a debt upon the policy, to be wiped off at a future day if the death has not intervened, whether the debt be estimated upon the careful method of Mr. Younger, or by the more empirical modes adopted by the Offices which advertise the system, deprives the Company of the chance of profit which alone can compensate them for the concurrent chance of loss; and enables the assured in effect to say to the Company, "Heads, I win; tails, you lose." Indeed, any debt whatever upon a policy is unsound, and subversive of the principles of assurance.

It may be worth while to add, as collateral to the subject, that what has been said does not affect the question of returning a bonus to the assured out of the resulting profit of a series of transactions, if such profit has arisen from prudent investment, gain of interest beyond that involved in the tables, economy in expenditure, or other like causes. But bonus ought never to be divided out of profit supposed to arise from a more favourable past experience than that for which the rates of premium have provided; and those Companies who boast that their claims have been less than the expected amount, unless they retain in their hands the money so saved, are congratulating themselves upon what is really a presage of future disaster, as far as the policies under observation are concerned. For the deaths that have been hitherto avoided are yet to happen, inasmuch as every death must occur at some time; and the fact that, in the early years of a Company, or of a series of policies, the mortality has been less than the tables would have allowed for, leads to the inference that in future years (as regards those policies) the deaths will exceed the expected amount to the same extent. And this, which theory leads us to anticipate, experience shows to be the actual result; for the advantage of medical selection ceases in a very few years, and the self-selection of the assured against the Company overbalances it—the good lives dropping their policies upon slight temptation, but the impaired lives keeping theirs in force at all risks, knowing them to be a property which, once lost, cannot be replaced. Reserve must, therefore, be made, not only for the expected mortality according to the "law," but also for the probability that the real future mortality will exceed it.

These may be "old-school" views; but I am inclined to think that the avidity of policy-holders for bonus, and the extraordinary measures of advertising now adopted, are evil results of the competition of the last few

years, and that it should be the part of your *Magazine* to maintain soundly conservative doctrines in life assurance. I crave your pardon for troubling you with these remarks, not having the honour of belonging to the Institute which your *Journal* represents with so much ability; but I believe you do not consider that qualification essential to correspondence with you.

I am, Sir,
Your obedient servant,
EDWARD W. BRABROOK, F.S.A.

27, *Regent Street,*
October, 1862.

P.S.—I add one line, to congratulate Mr. Younger and the Institute on his election as a Fellow; though we happen to differ on this occasion, a long acquaintance with him has led me to hold him and his merits in very high regard.

E. W. B.

ON MR. SAMUEL YOUNGER'S PLAN FOR THE ASSURANCE OF INVALID LIVES.

To the Editor of the Assurance Magazine.

Sir,—Will you allow me space for a few remarks on Mr. Younger's paper, in your last Number, on the assurance of diseased lives, as I think that gentleman rather needlessly complicates a comparatively simple matter.

A person desirous of making a specific annual payment or investment for the assurance of his life, has, obviously, various alternatives to choose from. He may assure the same sum over the whole period of life—a larger sum for but a part of his life—or a sum increasing by a certain ratio, as in most ordinary "Bonus" assurances. He may stipulate that the risk of the Office shall only commence a fixed number of years hence, as in a printed table now before me; or, as in the case detailed by Mr. Younger, it may be an understanding that the risk of the Office shall be for one (smaller) sum during an earlier period of the assurance, and for a larger sum thereafter—the *fixed* terms being, the amount of premium to be paid, the larger sum for which the Office is to be responsible, and the period when the larger risk is to commence; the *variable* term being the amount of risk during the earlier period. Of course, such a system can be carried out only within certain limits, and the amount of the smaller risk is obtained by the simple equation—

$$S=\frac{AM_x-\pi N_{x-1}}{M_x-M_{x+n}},$$

where S is the smaller amount required, π the premium agreed to be paid, reduced by its proportion of Office loading or commission, A the larger assurance agreed on, and n the number of years for which the risk of it is deferred.

By this we may work out the first example given by Mr. Younger, where $x=35$ (the *assumed* age of the diseased life), $n=34$, $A=100$, and $\pi=1{\cdot}7554$, the net annual premium by the Carlisle 4 per cent. table for a life assurance of £100, at age of 30. This premium is perfectly arbitrary,

and any other might have been taken for illustration—the result gives £82·4536 for the smaller sum to be assured, or £17·5464 less than the £100 to be finally covered. Mr. Younger proposes to deduct only £6. 12*s.*

The method has lately been made applicable to the assurance of diseased lives; the benefit being calculated at the increased age of the assured assumed by the Office, and the premium payable being that applicable to his real age. Certainly, more has been made of it than is warrantable, as it has been asserted to be a means of correction of any erroneous estimate of the deterioration of the life, although that estimate is strictly adhered to in the equation.

Mr. Younger now, however, proposes to deal in another way with the latter problem. Reasoning, from an assumed tendency of the medical officer to overrate a special risk, and from other causes, he thinks to correct the error by making his computation at the real age of the assured, deducting from the benefit, during the earlier period of the assurance, such an amount as is represented by the extra premium arbitrarily imposed by the exigencies of the case. The propriety of this view is, to say the least, very doubtful; but without stopping to discuss it, you will see that, assuming the extra premium payable for the whole of life, the deduction is to be calculated from the expression—

$$\frac{\pi N_{x-1}}{M_x - M_{x+n}},$$

π being the extra premium imposed; and dealing with Mr. Younger's first example, where $x=30$, $n=34$, and $\pi=\cdot35833$, the result gives the equivalent short assurance (by Carlisle 4 per cent.) to be £28·9808.

I am not very sure, however, of Mr. Younger's meaning, when he "suggests that the extra premium proposed to be charged should be left unpaid until the expiration of the period termed the expectation of life;" but assuming that his proposal is, to require payment from the assured, not for the whole of life, as recommended by his medical adviser, but for that term only during which the deduction from the sum assured is to be in force, and which he fixes at the tabular "Expectation," then, abiding by his first example, and the Carlisle 4 per cent. data, the present value of the 34 years' assurance, which he is to deduct from the benefit under the policy in question, is—

$$\frac{6\cdot6(M_{30}-M_{64})}{D_{30}}=1\cdot45684,$$

while the present value of but the first 34 payments of the premium, which he is to forego in respect of this risk, is—

$$\frac{\cdot358333(N_{29}-N_{63})}{D_{30}}=5\cdot89234,$$

or upwards of four times the first amount.

Such a discrepancy invites an inquiry into Mr. Younger's investigation, but I confess my inability to follow his reasoning. Assuming the truth of his own expression (1), as a correct estimate of the present value of the annuity forborne, he should, surely, have equated it, not with an assurance payable at death, but with one payable should the life fall within e years,

the early period he deals with; but I must leave it to himself, or any other of your contributors who will kindly give us some farther information on the point.

I am, Sir,

Your most obedient servant,

H. AMBROSE SMITH.

Aberdeen, 24th November, 1862.

ON MR. YOUNGER'S SCHEME FOR THE ASSURANCE OF DETERIORATED LIVES.

To the Editor of the Assurance Magazine.

SIR,—My attention has been directed to Mr. Younger's paper in your last Number (p. 268), on a scheme for the assurance of diseased and deteriorated lives, and I shall be glad to be allowed to make a few remarks on it.

The usual mode of dealing with deteriorated lives is to treat them as if so many years older than their actual age, the effect being the imposition of an additional amount of premium, which may or may not be commuted into a reversionary deduction. Mr. Younger is dissatisfied with this mode of treatment, and he propounds a scheme of his own, with a view to obviate his objections to that at present in use.

Mr. Younger's main position is, that the decision of "the medical officer [which regulates the number of years to be added to the age of the life, and, consequently, the amount of the extra premium], if incorrect, is far more likely to be in favour of the Company than otherwise"; and upon this, as a foundation, he erects his superstructure. Retaining the extra premium, imposed in accordance with the medical report, he proposes:—

1. That this premium shall be payable, not during the whole of the after-lifetime, but only during a portion of it.

2. That it shall not be payable even during the term in question, unless the assured die during the said term. And,

3. Commuting the premium thus curtailed and rendered contingent into a *whole-life* reversionary deduction, he proposes, finally, to restrict the duration of this deduction to the term already referred to.

Add to all this that Mr. Younger, in his valuations and commutation, uses the *real*, and not the *increased* age, and it will be strongly surmised that the concessions enumerated are vastly more than sufficient to meet a presumed *likelihood* that the medical report, *if incorrect*, errs in favour of the Office.

Let us examine a particular case. I take at random the first given by Mr. Younger—a life of 30, which is estimated by the medical officer as deteriorated to the extent of five years, and on which, accordingly, an additional premium of 7*s*. 2*d*.$=$·358333 per cent. is imposed. The present value of this premium (Carlisle 4 per cent.), as at age 35, is 6·1065. Mr. Younger, for this premium, substitutes a reversionary deduction of £6. 12*s*.$=$6·600, during the first 34 years—the present value of which is 1·7086. If, with Mr. Younger, we use the real age, the values come out 6·3970 and 1·4568 respectively, exhibiting a still greater disparity. The

medical report is thus, in fact, all but nullified; and it is hence easy to predict the consequences to an Office which should adopt Mr. Younger's scheme. Mr. Younger can hardly have instituted such a comparison as the above previous to propounding his scheme. If he had, I cannot believe that he would have considered the striking off of 77 per cent. from the penalty for deterioration, *all round*, the right thing to do in compensation of a *possible* error in a few cases, in favour of the Office.

The problem implied in Mr. Younger's scheme possesses interest as involving an allocation—I cannot call it payment—of premium which has not been heretofore considered. For that reason, therefore (as a useful exercise), and for others besides, I here give a solution of it.

Problem.

A temporary premium, π, to last t years, is exigible on (x), but is to be payable only if the life fail during the said term of t years. The *periodical* payment of such a premium being impracticable, it is to be commuted into a temporary reversionary deduction, V, for the same term of t years, from a whole life assurance on (x), which assurance may be one either already subsisting, or to be now constituted. Required V.

Solution.—A premium exigible as described, as a little consideration will serve to show, is the same thing, so far as pecuniary result is concerned, as the same premium, π, payable during t years, and subject to return, with compound interest, at the end of the term, if (x) shall be then alive.

The present value of the temporary premium is

$$\frac{\pi N_{x-1|t}}{D_x},$$

and the return being $\pi(1+r)(A)_t$, where $(A)_t$ is the tabular *amount* of an annuity certain of £1 for t years (the factor $1+r$ adapting it to the case of payment in advance), its present value is

$$\frac{\pi(1+r)(A)_t D_{x+t}}{D_x}, \text{ or } \frac{\pi[(A)_{t+1}-1]D_{x+t}}{D_x}.$$

Hence the value, subject to the return, is

$$\frac{\pi\{N_{x-1|t}-[(A)_{t+1}-1]D_{x+t}\}}{D_x}.$$

Equating this to

$$\frac{VM_{x|t}}{D_x},$$

the present value of the deduction, we get

$$V=\frac{\pi\{N_{x-1|t}-[(A)_{t+1}-1]D_{x+t}\}}{M_{x|t}}.$$

Applying this to the foregoing example, where $x=30$, $t=34$, and $\pi=\cdot358333$, we find for V, $9\cdot3784$. Mr. Younger has $6\cdot600$, the difference arising from his having, as already stated, determined the deduction as if it were to last for life, and *then* arbitrarily restricted its duration to 34 years.

Writing the above expression for the value of the premium, thus,

$$\frac{\pi\{N_{x-1}-[N_{x+t-1}+\overline{(A)_{t+1}-1}\,.\,D_{x+t}]\}}{D_x},$$

we see at once the nature and the effect of Mr. Younger's dealing with the medical report. The first portion, $\pi N_{x-1}\div D_x$, is the value of the premium arising from the medical officer's estimation of the risk, equal (if, with Mr. Younger, we erroneously used the *real* age) to 6·3970; and the remaining portion, $\pi\{N_{x+t-1}+[(A)_{t+1}-1]D_{x+t}\}\div D_x$, is the value of the portion of this premium that Mr. Younger abandons, equal to 4·3269. And this is irrespective of his subsequent curtailment of the duration of V.

Mr. Younger's expression for the value of the premium is reducible to that given above. It is needless to occupy space by showing it.*

It may interest some of the readers of the Magazine if I add a solution of the problem according to the old, and now almost disused, method.

That a premium will become due in respect of the nth year will depend on the concurrence of these two events; first, that (x) enters on that year, the probability of which is, $p_{x.n-1}$; and, secondly, that $(x+n-1)$ will not attain the age $x+t$, the probability of which is, $1-p_{x+n-1.t-n+1}$. The probability of the compound event therefore is,

$$p_{x.n-1}(1-p_{x+n-1.t-n+1})=p_{x.n-1}-p_{x.t}.$$

Multiplying by πv^{n-1} (the payments being due at the *commencement* of the year), and summing from 1 to t, we obtain for the value of the premium,

$$\pi\left(1+a_{\overline{x.t-1}|}-p_{x.t}\frac{1-v^t}{1-v}\right),$$

which may be put in either of the forms,

$$\pi\{1+a_{\overline{x.t-1}|}-p_{x.t}(1+r)(a)_t\},$$

$$\text{or }\ \pi\{1+a_{\overline{x.t-1}|}-p_{x.t}[(a)_{t-1}+1]\},$$

where $(a)_t$ and $(a)_{t-1}$ denote the tabular *present values* of annuities certain of £1 for t and $t-1$ years respectively.

To return for a moment to the scheme: Mr. Younger assumes, for the purposes of his investigation, that the incorrectness of the medical report is established, if the life reported on live over the term of years that he calls its "expectation." Mr. Younger knows theoretically that this is not true, for he admits as much. He nevertheless uses the hypothesis as if it were true, and we have seen the result. But I go a great deal further than a simple denial of the hypothesis, and I say that it is maintainable that no amount of *survivance* on the part of a particular life, although extending to the utmost limit of human existence, will suffice to prove that there was not, *at the date of the medical report*, sufficient ground for the relegation of that life to a class of a greater age than the age assigned by the date of its birth. If, in consequence of increased care (which is frequently engendered by delicacy of constitution), or from any other cause, the life in question should, contrary to anticipation, attain a good old age, the case must be

* Mr. Younger, like some other writers, denies himself the use of the very convenient symbol v, equivalent to $1\div(1+r)$, or $(1+r)^{-1}$. In consequence, his expressions are more cumbrous than they need have been.

considered as just one of those, the decision of which in favour of the Office, enables it to meet the claims arising on account of those that are decided against it. Having stood the risk of an adverse decision, the Office must not be called upon to surrender the consideration in respect of which it undertook that risk.

Mr. Younger intimates that one object he proposed to himself in the devising of his scheme was to exhibit a less value of the measure of deterioration than that exhibited by the existing method. The simplest way of doing this would have been to strike off a percentage from the result of the usual method, which way, moreover, would have had the further advantage of letting us know exactly what we were about. This, Mr. Younger's way of proceeding does not do. It is too complex for that. In fact, it is apt to remind one of the proceedings of the scientific tailors of Laputa, who, disdaining the use of a tape for the measuring of their customers, employed a sextant instead. The customers were, to be sure, very badly fitted. But what of that? The process was conducted on strictly scientific principles.

I am, Sir,

Yours most obediently,

Camden Town, 3rd December, 1862. P. GRAY.

PROFESSOR DE MORGAN'S QUERY ABOUT INTEREST ACCOUNTS.

To the Editor of the Assurance Magazine.

Sir,—Under the head of "Notes and Queries," in your *Magazine* for October, I find a notice by Professor De Morgan of a mismanaged interest account.

The Professor does not state the method his friend followed; so with your permission I shall endeavour to point out what has to be considered in making up an interest account of the nature described, on equitable principles—the course the debtor most likely followed—and the errors he fell into.

When money is lent at a certain rate of interest, no dates for the payment of such interest being mentioned, it is understood to be paid once a year; and if interest falls in arrear, and no penalty has been mentioned in the agreement, the least that can in equity be expected of the debtor is that he pay interest at the same rate on the arrears.

In framing an ordinary account current it is usual to calculate the interest on each Dr. and Cr. balance for the time it exists (within a year), keeping a note of the Dr. and Cr. interest, and to add or deduct, as the case may be, the difference at the end of each year. If interest be charged and allowed at the same rate, this method is the same as charging interest on each advance, and allowing interest on each payment, from the date it is made to the end of the year. A new accounting then commences, and the process is carried on from year to year during the continuance of the account. If there is but one payment made in each year, and that on the day the interest falls due, this process becomes similar to that described by Professor De Morgan—interest for the period is added to the principal, and the payment just made deducted; and this is the proper plan, whether the payment exceeds the interest or not. It would thus appear that the

balance of interest arising from the transactions of the year or other period agreed on should be added or deducted only at the end of such period. If the balance of interest is against the debtor, it then becomes part of the sum on which the next year's interest is calculated, unless paid by him, when he would lose the use of his money while the creditor gained it. In either case he pays *compound interest.*

Now the majority of accounts are made up once a year with a view to a settlement, or for reasons quite independent of the interest: so that many who are accustomed to make them up quite overlook the effect this has on the interest, if they ever think about it at all. When such a person is instructed to make up an account such as that described by the Professor, he would naturally proceed in the way he was accustomed to, quite overlooking the necessary balance at the end of each period at which interest *fell due*, balancing only at the end of the account, and thus charging merely simple interest; and the difference between this method and compounding interest yearly for a term of years could not fail to produce a startling result.

If the payments had been made regularly, the Professor's method of accounting was strictly correct; but if, as he states, they were irregular, it follows from what has been said that he should have balanced at the end of each year, not at the date of each irregular payment.

I think it was the first of these errors, that of charging simple interest only, that the debtor fell into when accounting with his friend and creditor; and afterwards, from not properly understanding Professor De Morgan's method, he applied it to an account on which there were several operations in a year, compounding interest at the date of each operation; or else, falling into the second error, he did not balance at the proper time, which may account for the solicitor requesting him to adhere to his original plan.

There appears to be a misprint in the debtor's reply to the Professor's question, as it is evident that whatever method gave him as a debtor *less* to pay would confer a similar benefit on any other debtor.

There are, I understand, certain accounts, as that between an agent who acts as a *quasi* banker and his client, which must be rendered before interest can be legally compounded. It is held that, until the account is rendered, the client is ignorant of its state, and is deprived of the power of paying the balance should it be against him; while, on the other hand, if the account is rendered and he does not pay the balance, which includes interest, it is held that he consents to the compounding. These considerations can have no weight in such a case as that described by the Professor, but it is just possible that the solicitor who objected to the compounding of interest was influenced by the practice arising from them, such practice being to allow simple interest only to the date at which the account is rendered.

I fear I have already trespassed on your space, but enclose a few practical illustrations of the subject, which you can insert if you think proper.

I am, Sir,
Your most obedient servant,

Edinburgh, Nov. 15th, 1862. A. H. T.

Illustrations.

1. *Simple Interest.*—A borrows £100 for 5 years, to be repaid at the end of that period, with interest at the rate of 5 per cent. per annum.

At the end of the 5 years he pays the principal, with £25 of interest; in all, £125.

2. *Compound Interest.*—B borrows £100 for 5 years, agreeing to pay interest *yearly*, at £5 per cent. per annum.

The interest which B pays is nominally the same as that paid by A (No. 1), viz., £25, but it is paid yearly, by which arrangement B loses the interest on the

1st year's interest for 4 years =	£1	0	0
2nd „ for 3 „ =	0	15	0
3rd „ for 2 „ =	0	10	0
4th „ for 1 „ =	0	5	0
In all,	£2	10	0

And his payments are equal to £127. 10*s*., viz.,—

Principal	£100	0	0
Interest, £25 and £2. 10*s*.	27	10	0
	£127	10	0

3. If, instead of repaying the principal in one sum at the end of the 5 years, the debtor pays £50 at the end of the second year, £25 at the end of the third year, and the remaining £25 at the end of the fifth year.

1st. If *simple* interest were payable the account would be stated thus:—

1855.	Jan. 1.	To sum lent				£100	0	0
1857.	„	To interest on £100—2 years,	£10	0	0			
„	„	By cash then received				50	0	0
		Balance				£50	0	0
1858.	„	To interest on £50—1 year,	£2	10	0			
„	„	By cash then received				25	0	0
		Balance				£25	0	0
1860.	„	To interest on £25—2 years,	£2	10	0			
		Add interest				15	0	0
		Balance, principal and interest				£40	0	0
„	„	By cash then received				40	0	0

The debtor thus pays—

Principal									£100	0	0
And interest, 5 years, on £100				£25	0	0					
Less on £50 for 3 years	£7	10	0								
„ on £25 for 2 years	2	10	0								
				10	0	0					
									15	0	0
									£115	0	0

2nd. If the interest were *compounded annually* the account would be as follows:—

1855. Jan. 1.	To sum lent		£100	0	0
1856. ,,	To interest for 1 year		5	0	0
			£105	0	0
1857. ,,	To interest, 1 year, on £105		5	5	0
			£110	5	0
,, ,,	By cash then received		50	0	0
	Balance		£60	5	0
1858. ,,	To interest, 1 year, on £60. 5*s*.		3	0	3
			£63	5	3
,, ,,	By cash then received		25	0	0
	Balance		£38	5	3
1859. ,,	To interest, 1 year, on £38. 5*s*. 3*d*.		1	18	3
			£40	3	6
1860. ,,	To interest, 1 year, on £40. 3*s*. 6*d*.		2	0	2
			£42	3	8
,, ,,	By cash then received		42	3	8
The debtor thus pays £50, £25, and £42. 3*s*. 8*d*.=			£117	3	8
Of which principal			100	0	0
	The balance being interest		£17	3	8
	Consisting of simple interest	£15 0 0			
	Interest on interest	2 3 8			
			17	3	8

In the above example interest is accumulated each year, but not paid. If it were paid each year the result would be the same—the debtor would lose £2. 3*s*. 8*d*. of interest on the interest so paid.

INSTITUTE OF ACTUARIES.

PROCEEDINGS OF THE INSTITUTE.

First Ordinary Meeting, Session 1862-63.—*Monday, 24th November,* 1862.

The President in the Chair.

The minutes of the annual general meeting were read and confirmed.

The Secretary announced various donations to the library.

Messrs. J. W. Holland and D. H. McGregor, duly nominated at the last ordinary meeting, were unanimously elected Associates of the Institute.

The following gentlemen were nominated for ballot at the next ordinary meeting :—

Official Associates.

J. H. Evens.
W. T. Hancock.

Associates.

Francis Addiscott.
Benjamin Cunningham.
Wm. Hughes.
T. H. Johnson.
A. E. Middleton.
Lieut.-Col. Oakes.
Chas. H. Ogbourne.
Henry Parminter.
Arthur Smither.
E. Waterhouse, B.A.

Mr. Tucker read a paper " On the proper mode of estimating the liabilities of Life Insurance Companies."

Thanks were voted to Mr. Tucker, and the meeting adjourned to Monday, 29th December, 1862.

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END OF VOL. X.

C. & E. Layton, Printers, 150, Fleet Street, London.

EAGLE INSURANCE COMPANY,

LONDON.

REPORT of the Directors for the Year ending 30th June, 1862, read at the Annual General Meeting, 8th August, 1862. DR. GUY *in the Chair.*

Another quinquennial term has passed away—the eleventh since the establishment of the Company—and the duty again devolves upon the Directors of reporting to the Proprietors as to the exact financial position of the Company, after a minute investigation of its liabilities, and a careful estimate of the value of its assets. But they will first briefly advert to the events of the past year, which it will be seen are not wholly devoid of interest.

The Surplus Fund Account exhibits the following particulars, viz.:—

SURPLUS FUND ACCOUNT.

Income of the Year ending 30th June, 1862.

		£ s. d.
Balance of Account, June 30th, 1861		786,546 6 3
Premiums on New Assurances	£24,374 18 5	
Ditto on Renewed ditto ..	284,910 18 0	
	309,285 16 5	
Interest from Investments ..	81,250 9 4	
		390,536 5 9
		£1,177,082 12 0

Charge on the Year.

		£ s. d.
Dividend to Proprietors		10,179 5 0
Claims on decease of Lives Assured	£182,278 5 6	
Additions to those under Participating Policies	24,816 3 1	
Policies surrendered	16,672 14 7	
Reassurances, New	6,219 10 0	
Ditto, Old	33,284 13 4	
	263,271 6 6	
Commission	11,456 2 6	
Medical Fees	999 12 8	
Income Tax	3,517 4 11	
Expenses of Management ..	12,224 1 11	
		291,468 8 6
Total Outlay		301,647 13 6
Addition to the Reserve Fund		239,392 4 2
Balance of Account, 30th June, 1862 .		636,042 14 4
		£1,177,082 12 0

Examined and approved,
THOMAS ALLEN,
WM. HENRY SMITH, Jun., } *Auditors.*

Here it will be seen that the total income of the year was £390,536. 5*s.* 9*d.*, and the total outlay £301,647. 13*s.* 6*d.* leaving a difference on the year in favour of the Company of £88,888. 12*s.* 3*d.*

The premiums on account of new assurances are £24,374. 18*s.* 5*d.*, of which sum £6,219. 10*s.* has been expended in reassurances. The claims amount to £207,094. 8*s.* 7*d.*, and are less than those of the previous year by upwards of £48,000. The expenses are about £1,000 more this year than they usually are, in consequence, partly, of the increased cost of the Company's new premises, and partly of the estimated depreciation in the value of the lease and furniture.

The Directors will now address themselves to the more important business of the Meeting, and will call the attention of the Proprietors to the following extract from the Report of the Actuary, which, after mature consideration, they have approved:—

"I have the honour to report that so much of the calculations as relates to the determination of the Company's assets and liabilities has been completed, and that the results are as follows:—

"On making a most careful examination of the Company's registers, it appesrs that there were in force on 30th June last, not less than 16,257 assurances of various descriptions—that the amount assured by them was £9,639,222, and that the annual premiums receivable in respect of them amounted to £307,376. The number and amount in each class are shown in the following Table, in which also the Participating is distinguished from the Non-participating portion :—

CLASS.	Participating Assurances.				Non-participating Assurances.		
	No. of Policies.	Sums Assured.	Existing Additions.	Annual Premiums.	No. of Policies.	Sums Assured.	Annual Premiums.
Single Life, Whole Term	11,546	5,992,410·0	360,956·5	199,302·31	2736	2,310,776·9	75,130·97
Endowment Assurances	324	95,482·0	3,105·3	4,189·81	316	75,866·6	3,274·48
Limited Payments	85	50,164·0	6,880·4	1,167·38	19	4,805·8	32·05
Increasing Premiums ..	76	28,869·0	1,700·1	1,100·38	214	145,100·9	6,360·95
Decreasing Premiums ..	4	9,000·0	1,753·6	125·83	1	400·0	8·40
Joint Lives	265	60,367·0	1,724·8	2,826·85	184	39,147·0	1,795·23
Last Survivor	27	38,448·0	5,824·8	728·28	73	62,855·3	1,276·70
Contingent	3	1,899·0	62·0	74·77	110	163,402·5	2,395·63
Endowments	..	..	..	..	121	25,505·9	1,007·63
Term Assurances	..	..	..	..	138	99,764·2	2,003·69
Special Assurances.....	..	..	..	..	15	52,950·0	
Climate Risks..........	..	..	..	..	..	..	4,574·72
	12,330	6,276,639·0	382,007·5	209,515·61	3927	2,980,575·1	97,860·45

	No.	Sums Assured.	Annual Premiums.
Total Participating and Non-participating........	16,257	£9,639,221·6	£307,376·06

"The calculations for the determination of the liabilities under these contracts were commenced some months since, and, by using very great efforts, have been completed in time for the Meeting, although the account was closed only on the 30th June last.

"Every possible precaution has been taken to insure strict accuracy, and to assign to each risk its proper value. The result of the whole will be seen in the following account, which exhibits, at one view, the value of the Company's assets and liabilities as they existed on the 30th June of this year:—

BALANCE SHEET.

LIABILITIES.		£	s.	d.
Interest due to Proprietors		3,480	12	9
Claims on decease of Lives Assured and additions thereto unpaid		56,983	2	5
Cash Bonus due to Policy-holders		683	4	8
Sundry Accounts..		4,454	9	4
Liability under Sums Assured, &c. (1862)		4,694,810	14	0
Proprietors' Fund	£198,312 10 0			
Surplus Fund, as per account	636,042 14 4			
		834,355	4	4
		£5,594,767	7	6

ASSETS.	£	s.	d.
Amount invested in Fixed Mortgages ..	1,087,719	18	6
Ditto ditto decreasing Mortgages	148,890	8	10
Ditto ditto Reversions	224,734	11	6
Ditto ditto Funded Securities	291,756	12	10
Ditto ditto temporary Securities	36,523	19	4
Current Interest on the above Investments..	25,353	15	3
Cash and Bills	28,965	1	3
Advanced on Security of the Company's Policies, &c.	110,333	19	8
Agents' Balances	24,011	3	1
Sundry Accounts	14,085	12	3
Value of Assurance Premiums (1862)	3,568,151	7	8
Value of Re-assurances	34,240	17	4
	£5,594,767	7	6

Examined and approved,

THOMAS ALLEN,
WM. HENRY SMITH, Jun., } *Auditors.*

"From this statement it appears that the liability under the Company's contracts is estimated at £4,694,810. 14*s.*, and that after making provision for this sum, for the Proprietors' capital, and for other items, there remains a balance on the surplus fund account of £636,042. 14*s.* 4*d.* This balance will, I consider, justify the division on the present occasion of £158,650, leaving £477,392. 14*s.* 4*d.* to accumulate in aid of future distributions. The Proprietors' share will be payable to them in October as usual. The Policyholders' portion will afford a *reversionary addition* to the sums assured, ranging, according to the age of the life assured, from twenty to seventy per cent. of the premiums paid since 1857; as soon as possible the amounts to be added in each case will be ascertained, and the usual notices forwarded; but some months must necessarily elapse before the whole can be dispatched."

Such is the result of the laborious investigation just made into the Company's affairs. To the Directors it is very satisfactory; and, they doubt not, that it will be equally so to the Proprietors. They therefore proceed at once to the remaining business of the Meeting. It is proposed that a Company having an annual income of about £80,000 shall merge in the Eagle on equitable terms, and one of the conditions stipulated for is that two of its Directors shall have seats at the Board of the Eagle. To this arrangement it is of course necessary that the Proprietors of the Eagle should give their assent: if they see fit to approve of it, it will be necessary to pass the following Resolution, and also to make provision for the remuneration of such new members of the Board in the event of the transfer taking place.

Here followed the Resolution, which, after an address from the Chairman, was, as well as the Report, unanimously approved and adopted. Some further routine business was then disposed of, and the Meeting separated, after passing the usual votes of thanks to the Directors and Officers.

The Directors of the Company is now constituted as follows, viz:—

Chairman—PHILIP ROSE, ESQ.

Deputy-Chairman—SIR JAMES BULLER EAST, BART., M.P.

CHARLES BISCHOFF, ESQ.
THOMAS BODDINGTON, ESQ.
CHARLES CHATFIELD, ESQ.
NATHANIEL GOULD, ESQ.
ROBERT ALEXANDER GRAY, ESQ.
WILLIAM AUGUSTUS GUY, M.D.
CHARLES THOMAS HOLCOMBE, ESQ.
RICHARD HARMAN LLOYD, ESQ.
JOSHUA LOCKWOOD, ESQ.
JAMES MURRAY, ESQ.
SIR W. G. OUSELEY, K.C.B., D.C.L.
W. ANDERSON PEACOCK, ESQ.
RALPH CHARLES PRICE, ESQ.
GEORGE RUSSELL, ESQ.
THOMAS G. SAMBROOKE, ESQ.
CAPT. L. S. TINDAL, R.N.
COL. CHAS. WETHERALL, K.C.T.
RIGHT HON. SIR JOHN YOUNG, BART.